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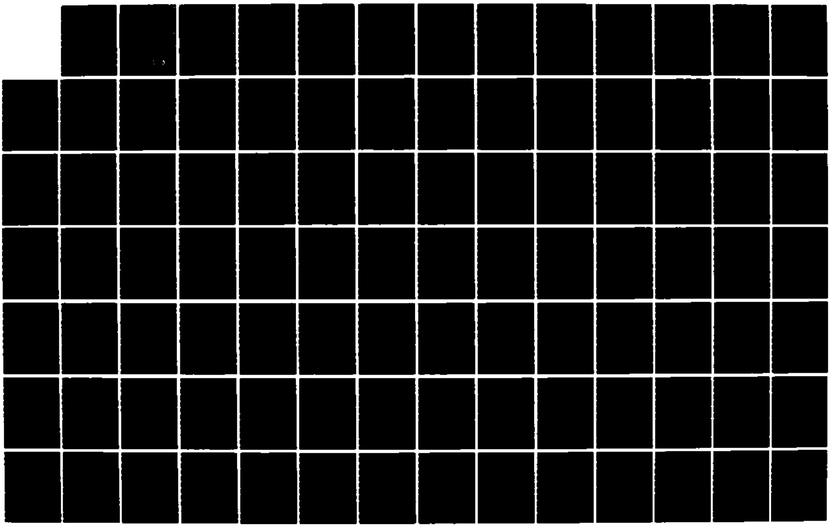

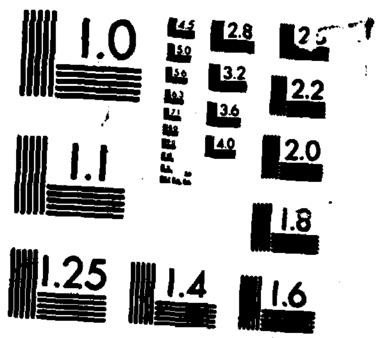

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ABSTRACT

Graphical Techniques for Maintenance Planning

Items subject to stochastic failure are candidates for preventive maintenance where often an item is replaced or restored to a good-as-new condition. For single component items, the optimal interval for this type of preventive maintenance might be determined via some mathematical model of replacement. Usually, these models require knowledge of the failure distribution as well as the costs of preventive and corrective maintenance. For most problems there is considerable uncertainty as to the appropriateness of a failure model. Bergman and others have espoused graphical techniques for the Age Replacement model which do not require specification of the failure model. ~~We~~ This dissertation proposes several graphical techniques for two other replacement models, Block Replacement and one ~~we~~ called Blind Replacement.

These techniques use historical data (a moderate amount) to estimate the optimal replacement interval. In the absence of such data the techniques here do not apply. With a view toward small-sample size statistical inference, we investigate the bias in graphical estimates of the optimal replacement interval, t*, for Age, Block, and Blind

Replacement. We find that, for small samples, techniques using the total time on test statistic may have positive bias. We propose a survivor time on test statistic which results in estimates that may have a negative bias. We recommend composite estimators in some cases to mitigate bias when sample sizes are small.

These graphical techniques provide only point estimates at t*. Accordingly, we propose two confidence bounds techniques using a bootstrap resampling approach. For Age and Blind Replacement, we show that these techniques in many cases produce useful confidence bounds for t*.

86-1

Graphical Techniques for Maintenance Planning

By

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CHAPTER I

INTRODUCTION AND SUMMARY

Introduction to Planned Maintenance

Maintenance strategies are sets of actions to be taken periodically on a repairable system in order to sustain its performance. An optimal strategy results in system performance at its best, as measured by a specified criterion. System performance can be classified into discrete categories of conditions, such as a two-state case of good and failed or a multi-state case with conditions between these extremes. For simple systems, that is single-component systems which either are in good condition or have failed, we can consider four basic maintenance strategies. Two of these strategies apply in the situation when the system's condition is known at every instant, while the other two apply when the condition is not known except by inspection.

The first basic strategy, used when the system's condition is known, is that of replacing (or restoring to a "good-as-new" condition) the item at failure or at age t, whichever occurs first. This policy is known as Age Replacement and requires knowledge of both the item's age and its condition. The second strategy uses

information on the item's condition, not its age; this policy is to replace the item at failure and at fixed points in time. Such a policy is known as Block Replacement; it avoids the expense of recording failure times and tracking age. However, it incurs the additional cost of possibly replacing relatively new items.

In the above two strategies, the only uncertainty is that of an item's "lifetime," i.e. the age when failure occurs. On the other hand, if an item's condition (either good or failed) is unknown except by periodic inspection, then there are two kinds of uncertainty: that of the item's lifetime and that of the item's condition between inspections. The third and fourth basic maintenance strategies address both of these uncertainties. The third strategy, which we call Maintenance Inspection, is to inspect the item periodically every t units of time and to replace only if there is a discovered failure. The fourth strategy, which we refer to as Blind Replacement, is to replace the item regardless of its condition every t units of time. This fourth strategy is prudent if the cost of replacement is less than the cost of inspection (as would be the case if an inspection were destructive to the item). Note that under both Maintenance Inspection and Blind Replacement, a failed unit will not be replaced at the time of failure. These strategies are to be used only under circumstances in which the failure of the item is not fatal to the system.

In this chapter, we provide a synopsis of the literature on three of these four strategies and summarize our findings and conclusions. Since Maintenance Inspection models are complex {cf. Brender (1963)} and do not appear amenable to graphical solution, we do not address them.

All of the three strategies that we address have a common structure known as a renewal process. In a renewal process, the instant in time, at which the item is made "good-as-new" by either repair or replacement, is a renewal point. A sequence of renewal points through time is a renewal process as shown in Figure 1.

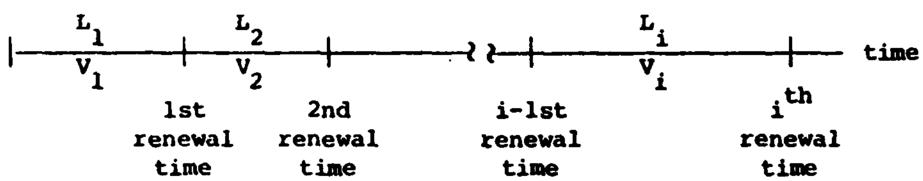

Fig. 1. A General Renewal Process

The times between renewal points, L_i for i=1, ... which are random variables, can be summed to obtain the time of the n^{th} renewal point, $S_n = \sum_{i=1}^{n} L_i$. Moreover, there is usually a cost (of repair, replacement, inspection, unavailability, etc.), V_i, incurred during $(S_{i-1}, S_i]$ the i^{th} renewal interval. This cost is also a random variable. The average cost per unit time during the interval $(0, S_n]$ equals the ratio $\sum_{i=1}^{n} V_i / \sum_{i=1}^{n} L_i$. Assuming that, in the two sequences of random variables $\{V_1, \ldots, V_i, \ldots\}$ and

$\{L_1, \ldots, L_i, \ldots\}$, (V_i, L_i) are independent and identically distributed, Ross (1983:78-79) proves that the long run, average cost per unit time is

$$\lim_{n\to\infty} \left\{ \frac{\sum_{i=1}^{n} V_i}{\sum_{i=1}^{n} L_i} \right\} = \frac{E\{V\}}{E\{L\}} \tag{1.1}$$

where $E\{V\} = E\{V_1\} = \ldots = E\{V_n\} = \ldots$, and $E\{L\} = E\{L_1\} = \ldots = E\{L_n\} = \ldots$. The solutions techniques for the probability models which we address all seek the value of the decision variable which minimizes this long run, average cost per unit time.

Age Replacement Model

Barlow and Proschan (1965) discuss Age Replacement using long run, average cost per unit time as the criterion. They presume a failure results in a cost, C_1, and that each replacement incurs a cost, C_2, where either a failure or a replacement results in a renewal of the system. They show that the long run, average cost, $C(t)$, for a policy of replacement at failure or at age t can be expressed as

$$C(t) = \frac{C_1\ F(t) + C_2\ R(t)}{{}_0\int^t R(x)\,dx}\ , \tag{1.2}$$

where F(t), the cumulative distribution function (cdf), is the probability that a failure will occur prior to or at age t, and R(t) is the probability that the item will survive beyond age t {i.e. R(t) = 1-F(t)}. The numerator of equation (1.2) is the expected cost per renewal interval, E{V}, while the denominator represents the expected length of a renewal interval, E{L}, given that lifetimes are truncated at age t, i.e.

$$E\{\min(L,t)\} = \int_0^t x\,dF(x) + t\int_t^\infty dF(x) = \int_0^t R(x)\,dx \ .$$

The quantity $\int_0^t R(x)\,dx = \mu(t)$ has a crucial role in Age Replacement. It is the expected operating time of an item which is withdrawn from service upon attaining age t. Clearly, $\mu(\infty)$ is the mean life (or mean time between failures given that there is no preventive replacement) while $\mu(t)$ is the "conditional mean life" (or mean time between maintenance).

Solution to Age Replacement When F(t) is Specified

Since R(t) = 1-F(t), equation (1.2) can be algebraically rewritten as

$$\tilde{C}(t) = \frac{C(t)}{C_1-C_2} = \frac{\dfrac{C_2}{C_1-C_2} + F(t)}{\int_0^t R(x)\,dx} \ . \qquad (1.3)$$

Although the criterion, $\tilde{C}(t)$, is recast in equation (1.3) as the long run, average standardized cost per unit time, a graphical clarity can be obtained as shown in Figure 2. If we consider the curve $\{x(t), y(t)\} = \{\mu(t), F(t)\}$ for $0 \le t \le \infty$, the slope, c, of any line $y = cx - C_2/(C_1-C_2)$ from the point $\{0, -C_2/(C_1-C_2)\}$ to $\{x(t), y(t)\}$ equals $\tilde{C}(t)$. The tangent line, as drawn in Figure 2, is the line with minimum slope and thus represents the optimal solution. The parameter value at the point of tangency, t^*, is the Age Replacement interval which achieves a minimum for $\tilde{C}(t)$. Therefore, t^* is an optimal interval with $\tilde{C}(t^*)$ equal to c^*, the slope of the tangent line. Note that there may be more than one point of tangency, in which case there are multiple solutions. If $\mu(\infty) < \infty$, there will always be at least one point of tangency as can be seen from the geometry in Figure 2. A nice feature of this graphical approach is that it avoids regularity conditions for unique optimum solutions such as F(t) belonging to the Increasing Failure Rate (IFR) class of distributions. We remark that the curve $\{x(t), y(t)\}$ is increasing (that is to say nondecreasing) in t because both F(t) and $\mu(t)$ are increasing functions in t.

Solution to Age Replacement
When F(t) is Estimated

In order to use equation (1.3) and its graphical representation for determining t^*, the failure distribution, F(t), must be specified. Often in practice, F(t) is

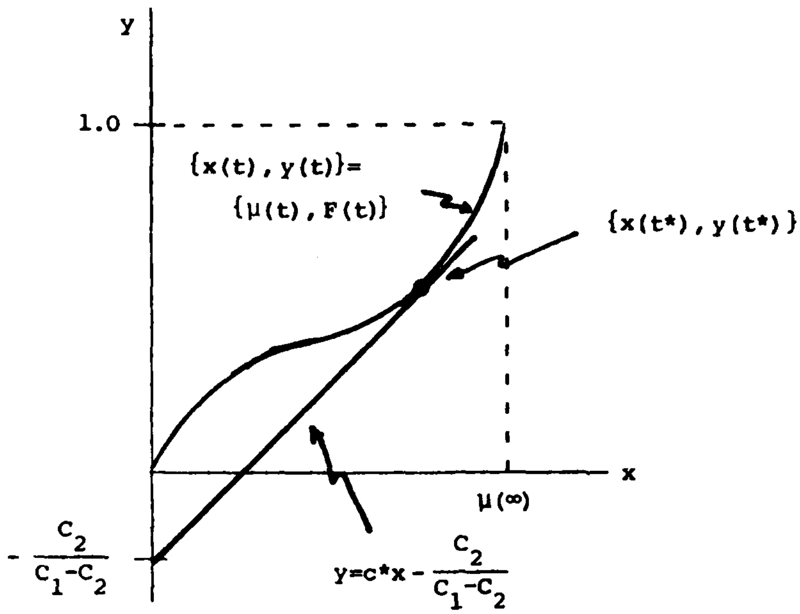

Fig. 2. Graphical Representation of Age Replacement

not known, and neither is it reasonable to assume a lifetime distribution. Arunkumar (1972) addresses this problem using ordered failure times to arrive at an empirical failure distribution and at an analytical solution for t^*. Bergman (1977a) develops a graphical technique for estimating t^* using the "total time on test statistic."

Bergman's technique supposes that there are n observed lifetimes $\{t_1, t_2, \ldots, t_n\}$ of an item. These are "free-flowing" (unrestricted) life lengths from a complete life test obtained from the laboratory instead of from truncated field data. These n lifetimes can be ordered as $\{t_{(1)} \leq t_{(2)} \leq \cdots \leq t_{(n)}\}$, and the "total time on test," $T(t_{(i)})$, through the i^{th} failure time, $t_{(i)}$, can be calculated as

$$T(t_{(i)}) = \sum_{j=1}^{i} (n-j+1)\{t_{(j)} - t_{(j-1)}\}$$

where $t_0 = 0$. The ratio $T(t_{(i)})/T(t_{(n)}) = U_i$ is the scaled total time on test at age $t_{(i)}$, and a plot of U_i versus i/n is known as a total time on test (TTT) plot. As we discuss below, Bergman's idea is to change the scale of the denominator of equation (1.3) so as to take advantage of the geometric properties of the total time on test plot. He then approximates this scaled denominator by U_i and estimates the failure distribution, $F(t_{(i)})$, in the numerator of equation (1.3) by i/n.

Bergman's technique is illustrated in Figure 3 for a sample of size four wherein we plot the point $-C_2/(C_1-C_2)$ on the horizontal axis and construct a tangent to the TTT plot passing through this point. We remark that Figure 3 is a reflection of Figure 2 rescaled about the line $y = x$. We present Bergman's technique as it is in Figure 3 because others have used this same graphic portrayal [cf. Barlow (1978)]. For reasons discussed below, we prefer the graphical representation in Figure 2.

Bergman's technique involves several approximations. First, he estimates the theoretical failure distribution by an empirical distribution, F_n. Since $\lim_{n\to\infty} F_n = F$, a finite number of observations results in an approximation of F that improves as the number of observations is increased. With ample observations, then,

equation (1.3) can be approximated as

$$\tilde{C}(t) \simeq \tilde{C}_n(t) = \frac{\frac{C_2}{C_1 - C_2} + F_n(t)}{\int_0^t R_n(x)\,dx} . \tag{1.4}$$

Bergman (1977a) states that an estimate of t*, the minimizer of equation (1.4), may be found among the n ordered lifetimes $\{t_{(1)} \leq \ldots \leq t_{(n)}\}$; "thus to estimate the optimal age replacement interval it is enough to find the index j for which

$$\tilde{C}_n(t_{(j)}) = \frac{\frac{C_2}{C_1 - C_2} + F_n(t_{(j)})}{\int_0^{t_{(j)}} R_n(x)\,dx} \tag{1.5}$$

is a minimum." Finally, Bergman estimates the empirical failure distribution using the plotting position "i/n" and the conditional mean life using the total time on test statistic, $T(t_{(i)})$, divided by n. A plotting position is a representation of the empirical cumulative probability, $F_n(t_{(i)})$, of the i^{th} ordered observation, $t_{(i)}$. We discuss the plotting position of i/n in greater detail later. Thus, using these two estimates, equation (1.5) is further approximated as

$$\tilde{C}_n(t_{(i)}) \simeq \left\{\frac{n}{T(t_{(i)})}\right\} \left\{\frac{C_2}{C_1 - C_2} + \frac{i}{n}\right\} . \tag{1.6}$$

Since $U_i = T(t_{(i)})/T(t_{(n)})$,

$$\tilde{C}_n(t_i) \approx \left\{\frac{n}{T(t_n)}\right\} \left\{\frac{\frac{C_2}{C_1-C_2} + \frac{i}{n}}{U_i}\right\} . \tag{1.7}$$

In lieu of minimizing equation (1.7) Bergman maximizes its reciprocal, i.e. he maximizes the ratio

$$\frac{U_i}{\frac{C_2}{C_1-C_2} + \frac{i}{n}}$$

since the term $n/T(t_{(n)})$ is constant. The geometric idea is that this ratio, which is the slope of any line passing through the point $\{-C_2/(C_1-C_2),0\}$ and the TTT plot, is proportional to $\tilde{C}_n(t_{(i)})^{-1}$. In order to minimize $\tilde{C}_n(t_{(i)})$, we need only maximize this slope. The maximum slope is attained by a line tangent to the TTT plot passing through the point $\{-C_2/(C_1-C_2),0\}$ as in Figure 3. In Figure 3, the numerator i in the value i/n corresponding to the point of tangency is the index of $t_{(i)}$, an estimate of the optimal replacement interval, t* (in Figure 3 this estimate is the failure time of the 3rd failure).

Bergman's technique is a significant contribution to the area of maintenance planning. With this method, maintenance analysts can avoid the necessity of goodness-of-fit testing and the inherent risk of incorrectly specifying a theoretical failure distribution. Although our

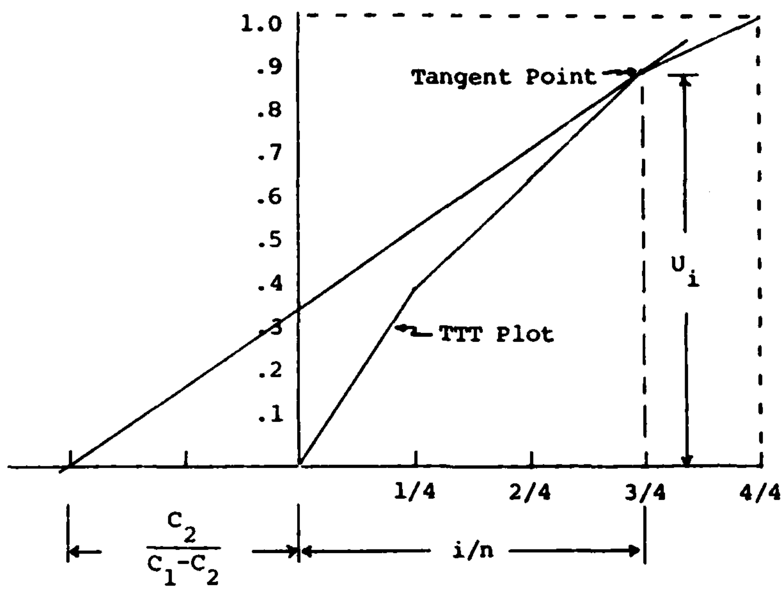

Fig. 3. Bergman's Graphical Technique

graphical approach, in Figure 2, might appear at first to differ from Bergman's approach, each is a reflection image of the other (with different scaling) which leads to the same result. Thus, our rationale, embodied in Figure 2, is only a refinement of Bergman's graphical argument.

From a practical point of view, Bergman's method is appealing because it lends itself to sensitivity analysis for cost values. Since management may be more comfortable with a range of cost values rather than point values, tangent lines drawn through the end points of a range of costs will display cost sensitivity in terms of a range of the estimated optimal Age Replacement interval.

Plotting Position and the Total Time on Test Statistic

Bergman's technique uses the plotting position i/n, known as the California position, to estimate the failure distribution at $t_{(i)}$. The California position is perhaps the most frequently encountered representation of F_n, and as we show below, it is intrinsic to the total time on test statistic. There are other plotting positions such as i/(n+1), the mean position; (i-1)/(n-1), the modal position; and (i-1/2)/n, known as Hazen's position. Weibull (1939a,b) recommended the mean position since, for the i^{th} order statistic $X_{(i)}$ (a random variable), the expectation of the cumulative probability at the i^{th} order statistic, $E\{F(X_{(i)})\}$, is i/(n+1). This result holds regardless of the underlying distribution generating ordered observations. Harter (1984) provided a comprehensive review of the plotting position literature and noted that

> ... much of the disagreement and confusion as to the choice of plotting positions is due to the fact that the cdf at the expected value of the ith order statistic is not equal to the expected value of the cdf at the ith order statistic, i.e.
> $$F[E\{X_{(i)}\} \neq E\{F(X_{(i)})\}$$
> except in the case of the uniform distribution $F(x) = x,\ 0 < x < 1$.

In Chapter II, we investigate the question of bias in Bergman's technique via Monte Carlo simulation and conclude that, for some common distributions, it produces biased estimates of t*. There may be several reasons for this bias. For example, Cunnane (1978) shows that the

California plotting position among others is biased for several common probability plots. Such a bias would affect Bergman's technique both in its estimate of $F(t)$ and of $\mu(t)$.

Any bias in $\mu(t)$ would stem from the total time of test statistic which is directly related to the empirical failure distribution, $F_n(t)$. The standard convention is to represent $F_n(t)$ as a right continuous step function (as we show in Figure 4 for the uniform distribution and its California position representation). The plotting position determines the step size at $t = t_{(i)}$ and the starting value $F_n(t=0)$. In Figure 4, we depict the steps occurring at $E\{t_{(i)}\} = i/(n+1)$ for ease of illustration, although in practice this would be an unlikely realization. The standard empirical survival distribution, $R_n(t)$, on the other hand, equals $1-F_n(t)$ as shown in Figure 5. Regardless of the shape of the distribution, an estimate of $\mu(t_{(i)})$, which is the area under $R_n(t_{(i)})$, can be calculated for the California plotting position as

$$\frac{n}{n}\left\{t_{(1)}\right\} + \frac{n-1}{n}\left\{t_{(2)}-t_{(1)}\right\} + \ldots + \frac{n-i+1}{n}\left\{t_{(i)}-t_{(i-1)}\right\}$$

$$= \frac{\sum_{j=1}^{i}(n-j+1)\left\{t_{(j)}-t_{(j-1)}\right\}}{n}$$

where $t_0 = 0$. Note that the numerator is the total time on test statistic, $T(t_{(i)})$. It follows, then, that any bias

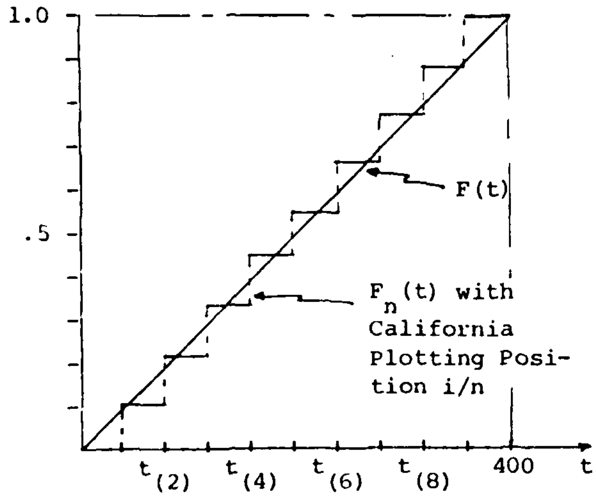

Fig. 4. Plot of F(t) and $F_n(t)$ for Sample of Size 9 from a Uniform (0,400) Distribution

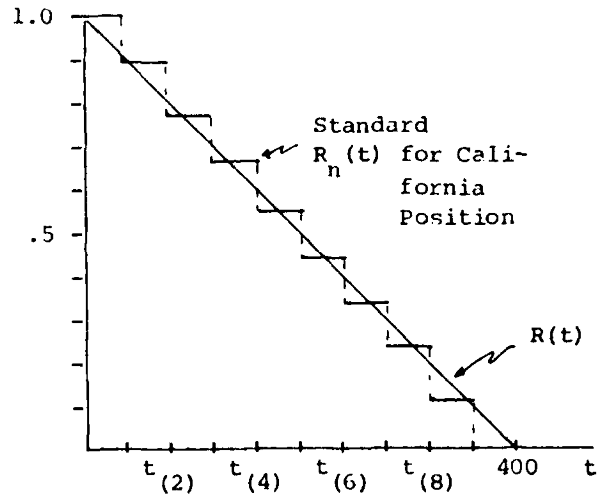

Fig. 5. Plot of R(t) and $R_n(t)$ for Sample of Size 9 from a Uniform (0,400) Distribution

in the empirical failure distribution due to the California plotting position will also bias the estimate of mean life, $\mu(t)$, through the total time on test statistic.

As an alternative to the total time on test statistic, we propose another statistic which we call the "survivor time on test statistic," $S(t_{(i)})$. This statistic is based on the time on test accumulated by surviving items which live beyond the age $t_{(i)}$. It is computed, at the i^{th} failure time, $t_{(i)}$, as

$$S(t_{(i)}) = \sum_{j=1}^{i} (n-j) \left\{ t_{(j)} - t_{(j-1)} \right\}$$

where $t_0 = 0$. In Chapter II, we show that a composite estimator of t^* using both $T(t_{(i)})$ and $S(t_{(i)})$ may produce less bias under some circumstances than Bergman's estimator.

Block Replacement Model

Barlow and Proschan (1965) also discuss a probability model for the second maintenance strategy where an item is replaced at failure as well as at fixed calendar time intervals of length t regardless of age. Their objective is to minimize long run, average cost per unit time where, as in their Age Replacement model, C_1 denotes the cost of a failure and C_2 represents the cost to replace the item. In Figure 6, we picture this renewal process with renewals occurring at replacement every t units of

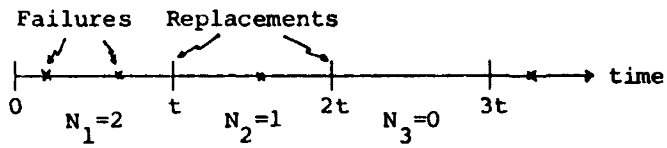

Fig. 6. A Block Replacement Renewal Process

time. Thus, in this model, the expected renewal length, E{L} in equation (1.1), equals t. To obtain V_i, the cost of the i^{th} renewal interval, we can multiply C_1 by the number of failures, N_i, in the i^{th} renewal interval $((i-1)t, it]$ and then add C_2 in order to account for the cost of the single replacement. The expected cost during a typical renewal interval, E{V}, can be expressed as

$$E\{V\} = C_1\ M(t) + C_2$$

where $M(t) = E\{N_i\}$, the renewal function [cf. Ross (1980: 228-230)]. Thus, assuming the replacement process will continue indefinitely, the long run, average cost per unit time, B(t), for a Block Replacement policy is a function of the replacement interval, t, and can be expressed as

$$B(t) = \frac{E\{V\}}{E\{L\}} = \frac{C_1\ M(t) + C_2}{t} \tag{1.8}$$

As an aid to a graphical representation, we can recast equation (1.8) in terms of the long run, average <u>standardized</u> cost per unit time, i.e.

$$\tilde{B}(t) = \frac{B(t)}{C_1} = \frac{M(t) + \frac{C_2}{C_1}}{t} \qquad (1.9)$$

Solution of Block Replacement When M(t) is Specified

If M(t) is specified, then analytical or numerical solution techniques can be used to derive the optimal t* which minimizes B(t). However, equation (1.9) has a useful geometric interpretation when we plot y = M(t) versus t as depicted in Figure 7. The renewal function, M(t), determines the shape of this plot which can be shown to be nondecreasing. If we consider the curve {t, y=M(t)} for $0 \leq t \leq \infty$, the slope, b, of any line $y = bt - C_2/C_1$ from the point $\{0, -C_2/C_1\}$ to {t, M(t)} equals $\tilde{B}(t)$. The tangent line, as drawn in Figure 7, is the line with minimum slope and thus represents the optimal solution. The value t* at the point of tangency is the Block Replacement interval which achieves a minimum for $\tilde{B}(t)$. Therefore, t* is an optimal interval with $\tilde{B}(t^*)$ equal to b*, the slope of the tangent line. The utility of this graphical representation centers on a visual means of sensitivity analysis for various values of C_2/C_1. With a straight edge, we can easily estimate t* for different cost ratios.

Depending on the shape of the renewal function, a finite t* may not exist. In the case of exponential failures, for example, the renewal function is linear, and

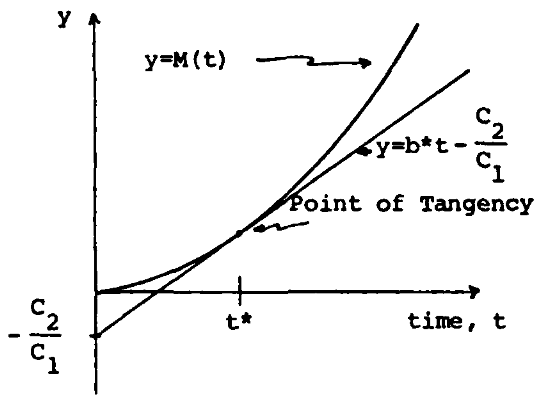

Fig. 7. Graphical Representation of Block Replacement

thus the point of tangency occurs at infinity as long as $C_2/C_1 > 0$.

Solution to Block Replacement When M(t) is Estimated

In Chapter III, we estimate the optimal Block Replacement interval using the above graphical technique and an empirical estimate of the renewal function. This empirical renewal function, $M_n(t)$, is developed from the superposition of a number of renewal sample paths. Suppose we have identical components on test in n positions, and we replace these components immediately upon failure. The sequence of failure times, N_i, in say the i^{th} component position, represents a sample path of the renewal process. As in Figure 8, we can superimpose the n sample paths on each other so as to constitute a superposition process [cf. Karlin and Taylor (1975)]. The superimposed number of failures $\sum_{i=1}^{n} N_i(t)$, form the series $\{Z_{(1)}, Z_{(2)}, \ldots, Z_{(m)}\}$ where the last superimposed

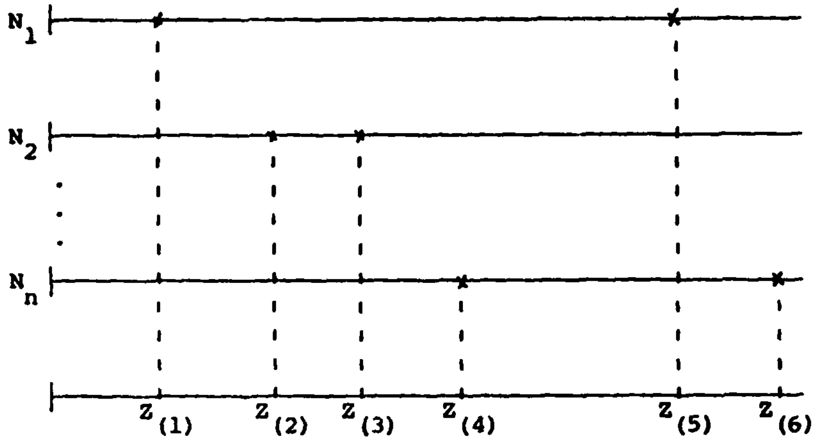

Fig. 8. Superposition of n Renewal Processes

failure time, $Z_{(m)}$, could be determined by some life test stopping rule. Then, by the Strong Law of Large Numbers,

$$\lim_{n \to \infty} \left\{ \frac{\sum_{i=1}^{n} N_i(t)}{n} \right\} = M(t) \ .$$

There are two methods of stopping this life test, i.e., we stop at a specified time, t_0, or stop after a certain (say r) number of failures have occurred at each component position. In the first case, both the number of superimposed failures and their failure times are random variables. Moreover, the number of superimposed failures could equal zero. In the latter case, only the superimposed failure times are random variables because the number of failures is fixed at $m = nr$. In the interest of ensuring some superimposed failure times, we choose

to stop after r failures have occurred in each component position.

The graphical solution technique involves estimating t^* by one of the superimposed failure times, $z_{(j)}$, and approximating the renewal function at $z_{(j)}$ as

$$M(z_{(j)}) \approx M_n(z_{(j)}) = \frac{j}{n} .$$

Our strategy is to plot $y = M_n(z_{(j)})$ as in Figure 7 and estimate t^* from the point of tangency. In Chapter III, we investigate (via Monte Carlo simulation) the accuracy and bias of our estimator as well as the relative error in the objective function due to the estimate. We conclude that a large number (greater than 30) of component positions may be necessary in order to reduce the likelihood of making a substantial relative error in $B(t^*)$ due to our estimator. If the underlying distribution is exponential and the cost ratio $-C_2/C_1$ is small (close to the origin), there is a large probability of incorrectly estimating a finite t^*; however, the relative error in $B(t^*)$ will tend to be small. We propose a data augmentation technique for use with scant data but show that it may adversely affect our estimate when the number of component positions is large.

Blind Replacement Model

Radner and Jorgenson (1962) formulate a model where a replacement occurs at intervals of time $t + K$

regardless of the system's state, i.e. if we are uncertain about the system's condition, we replace at periodic time intervals rather than inspect and replace as necessary. This model is fundamentally different from Age and Block Replacement in that we do not replace at failure (an impossible action if we do not know the instant of failure). This Blind Replacement strategy is reasonable only if the cost of an inspection is greater than the cost of replacement or if an inspection will destroy or consume the item. Otherwise, we would opt for a Maintenance Inspection strategy.

Radner and Jorgenson suppose that each replacement takes K units of time so that the system is renewed every $t + K$ units of time as shown in Figure 9. Thus, the expected renewal interval length, $E\{L\}$ from equation (1.1), equals $t + K$. They describe, as their criterion, "goodtime" (i.e. availability), G_i during the i^{th} renewal interval. Since $E\{V\} = E\{G_i\} = \int_0^t R(x)dx$, the long run, average availability per unit time is

$$G(t) = \frac{E\{V\}}{E\{L\}} = \frac{\int_0^t R(x)\,dx}{t + K} \qquad (1.10)$$

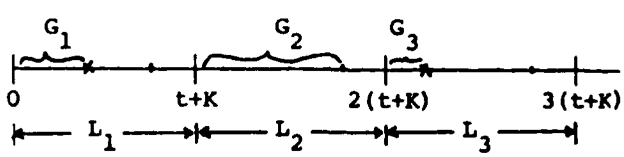

Fig. 9. A Blind Replacement Renewal Process

Solution to Blind Replacement When F(x) is Specified

A graphical solution to Blind Replacement can be found by plotting the curve $\{t, y = \int_0^t R(x)dx\}$ for $0 \leq t \leq \infty$ as in Figure 10. The slope, g, of any line $y = g(t+K)$ from the point $\{-K,0\}$ to $\{t, \int_0^t R(x)dx\}$ equals the value of the objective function G(t) at t. The tangent line, as drawn in Figure 10, is the line with maximum slope and thus represents the optimum solution. The value t* at the point of tangency is the Blind Replacement interval which achieves a maximum for G(t). Therefore, t* is an optimal interval with G(t*) equal to g*, the slope of the tangent line.

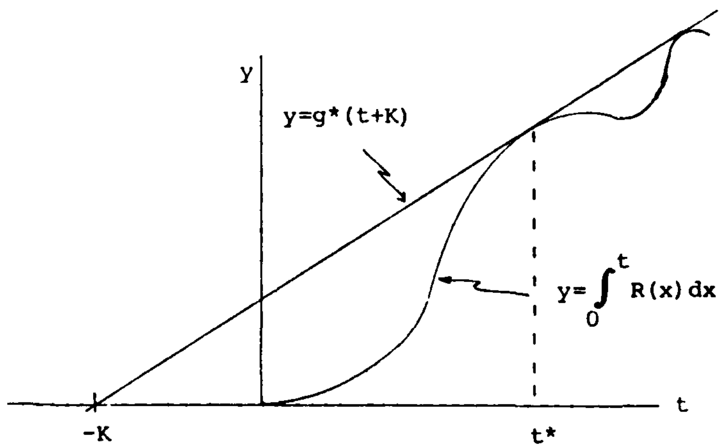

Fig. 10. Graphical Representation of Blind Replacement

This graphical approach is similar to Bergman's method shown earlier in Figure 3. However, there are several minor differences. First, with this approach, it is not necessary to scale the conditional mean life, $\int_0^t R(x)dx)$. Also, the value of G(t*) can be obtained

directly from the picture as the slope, g^*. Finally, the optimal Blind Replacement interval t^* can be read directly on the abscissa.

Solution to Blind Replacement When F(t) is Estimated

We suggest estimating the optimal Blind Replacement interval t^* by one of the observed failure times and approximating the conditional mean life by the total time on test statistic divided by n. In Chapter IV, we report on the results of Monte Carlo simulations of this technique for several DFR and IFR Weibull distributions. We conclude that the total time on test statistic produces positively biased estimates of t^* for a small to moderate replacement time, K. If we use the survivor time on test statistic in lieu of the total time on test statistic, then the technique produces negatively biased estimates of t^* for moderate to large K. As with Age Replacement, we recommend a composite estimator for Blind Replacement in order to mitigate bias.

Confidence Intervals for Maintenance Planning

In the graphical solutions to the three maintenance strategies discussed, we have only point estimates of the optimal replacement interval t^*. We show, in Chapters II, III, and IV, that these graphical techniques are biased and that the degree of bias diminishes as the sample size increases. A useful measure, in addition to a biased point

estimate of t*, is a (1-α) 100 percent confidence interval for t*. In Chapter V, we propose two confidence bounds techniques based on Efron's (1982) bootstrap resampling approach. We show, via Monte Carlo simulation for Age and Blind Replacement, that these techniques can produce, in many cases, useful confidence bounds. However, for Age Replacement, the confidence bounds may not contain t* when there are very large or very small differences between the cost of failure and the cost of replacement.

Summary

Statistical inference issues are of concern whenever samples are used to infer the characteristics of a population. Since graphical solution techniques for the above-mentioned maintenance models employ estimates of the survival probability and the conditional mean life based on samples, the statistical characteristics of these estimates are of interest.

We investigate the statistical inference issues of graphical solutions for Age Replacement, Block Replacement, and Blind Replacement in Chapters II, III, and IV respectively. In many (but not all) cases, we observe a positive bias in the estimators for t* for all three models. This positive bias in Age and Blind Replacement can be offset in some cases with a composite estimator based on the total time on test statistic and the survivor time on test statistic.

Since these graphical techniques provide only point estimates of the optimal replacement interval, we are also interested in a confidence interval methodology. In Chapter V, we propose such a methodology using a bootstrap resampling approach and demonstrate its utility for both the Age Replacement and Blind Replacement models.

CHAPTER II

AGE REPLACEMENT

In an Age Replacement maintenance scenario, a single-part item is replaced or renewed to good-as-new condition at failure or at age t whichever event occurs first. Barlow and Proschan (1965) provide a comprehensive synopsis of the literature relating to Age Replacement and a concise survey of useful renewal theory. Their development of a transcendental solution for the optimal Age Replacement interval, t*, presupposes that the costs of failure (C_1) and replacement (C_2) are known with $C_1 > C_2$, that the survival distribution belongs to the Increasing Failure Rate (IFR) class, and that any failure (or life until age t) is instantly replaced the moment it occurs. This last assumption requires instantaneous knowledge of condition and age, information which may be very expensive to obtain. This expense is not considered in their formulation.

Assuming F(t) is differentiable in equation (1.2), Barlow and Proschan show that the t* which minimizes long run, average cost per unit time must satisfy

$$\left\{\frac{f(t^*)}{R(t^*)} \int_0^{t^*} R(x)\,dx\right\} - F(t^*) = \frac{C_2}{C_1 - C_2} \qquad (2.1)$$

where f(t) is the derivative of F(t). So long as F(t) is IFR, they note that equation (2.1) implies $t^* > \{C_2/C_1\} \int_0^{\infty} R(x)\,dx$, i.e. we would never want to schedule a preventive replacement prior to this fraction of the mean life (which is not the same as the conditional life). They also remark that t* will be finite if the ratio of the standard deviation to the mean of the survival distribution is less than $\{1-(C_2/C_1)\}$.

If, on the other hand, the survival distribution belongs to the Decreasing Failure Rate (DFR) class where the instantaneous probability of failure decreases with age, then Barlow and Proschan point out that Age Replacement is inappropriate since a used item has greater expected remaining life than a new item and, hence, there is no incentive to replace. As a special case, the exponential distribution is both IFR and DFR; its failure rate neither increases nor decreases but rather is constant. We can easily show that equation (2.1) has no solution if the survival distribution is exponential. Therefore, we would elect to forego preventive maintenance in this case and establish the optimal Age Replacement interval at infinity.

The IFR class contains many different distributions including the truncated normal and most of the many

Weibull and gamma distributions. For these three types of distributions, Glasser (1967) provides mnemonic graphs based on equation (2.1) to aid the practitioner in determining the optimal Age Replacement interval. Of course, an analyst could solve equation (2.1) via some search technique, or for that matter, search techniques could be used directly on the objective function, in equation (1.2), should $F(t)$ not be differentiable. Nonetheless, the failure distribution must be specified a priori. In a situation with scant failure data, however, there may be some uncertainty as to the appropriate failure distribution. With this problem in mind, Arunkumar (1972) considers a nonparametric approach to estimating $F(t)$.

Arunkumar (1972) introduces the concept of an empiric approximation to the survival distribution in Age Replacement without recommending a plotting position. He shows that an estimator of the optimal Age Replacement interval, based on some empirical survival distribution, is strongly consistent. In an applied innovation, Bergman (1977a) expands on this idea. He uses an empirical cdf based on the California plotting position as an approximation to the failure distribution. As we discuss in Chapter I, this approximation leads to the total time on test statistic which, when divided by n, approximates the conditional mean life. In this chapter, we consider the rapidity by which Bergman's method, which we outlined in

Chapter I, converges to the optimal Age Replacement interval.

Although we focus on Barlow and Proschan's Age Replacement model, there are variations on this model which also may be of interest to the maintenance community in terms of graphical solutions. We do not pursue this issue here, but rather, we detail several variations to indicate a potential for further research. For example, Scheaffer (1971) supposes that the cost function, C(t), includes a term which increases with an item's age (to account for increasing replacement cost "due to depreciation or wear"). His results require explicit knowledge of the relationship between cost and age, which may be exceedingly difficult to obtain in practice. Weiss (1956) formulates an Age Replacement model which includes the time and cost of repair as well as the cost of failure and replacement. He indicates that an optimal solution may not always exist.

Graphical Solutions for Age Replacement

The total time on test statistic is used by Bergman (1977a) to develop a graphical solution for Age Replacement. The total time on test statistic,

$$T(t_{(i)}) = \sum_{j=1}^{i} \{t_{(j)}-t_{(j-1)}\}\{n-j+1\} ,$$

when divided by the total time on test, $T(t_{(n)})$, is the scaled total time on test, U_i, at age $t_{(i)}$. The plot of

U_i versus i/n is the total time on test (TTT) plot. Barlow and Campo (1975) point out that, as $n \to \infty$, an IFR distribution results in a concave TTT plot on the interval (0,1), whereas a distribution from the DFR class will have a convex plot. An exponential life distribution (which belongs to both the IFR and DFR classes) will have a linear TTT plot passing through the origin at a 45 degree angle, i.e. the plot is both concave and convex. An empirical life distribution need not belong to either the IFR or DFR class, and thus, its TTT plot would not necessarily be concave or convex. Barlow (1979) further describes properties of TTT plots and shows that concave (convex) TTT plots imply IFR (DFR) survival distributions. If the survival distribution belongs to the Increasing Failure Rate Average (IFRA) class or to the Decreasing Failure Rate Average (DFRA) class, then he proves that the TTT plot will be anti-starshaped for IFRA distributions and starshaped for DFRA distributions. A starshaped TTT plot is one that is on or below a linear TTT plot passing through the origin at a 45 degree angle but that is not necessarily convex. Conversely, an anti-starshaped TTT plot is on or above a 45 degree line through the origin but is not necessarily concave. Although an anti-starshaped TTT plot is not sufficient evidence that the life distribution is IFRA, Barlow (1979) remarks that this is sufficient to conclude that a survival distribution "is at least NBUE [New Better than Used in Expectation,

a class of distributions which includes the IFRA class] if not IFRA."

Berman's method employs the TTT plot as a scaled representation of the conditional mean life, i.e., U_i is proportional to $T(t_{(i)})/n$. Hence, the $t_{(j)}$ which maximizes the ratio $U_i/\{(C_2/(C_1-C_2)) + (i/n)\}$ will also maximize the ratio $(T(t_{(i)})/n)/\{(C_2/(C_1-C_2)) + (i/n)\}$. As Bergman (1977a) and others note, if $t_{(j)} = t_{(n)}$ then "it seems reasonable to estimate the age replacement interval as infinity," i.e. $\hat{t}^* = \infty$. This rationale is consistent with the Age Replacement solution, $t^* = \infty$, should the underlying survival distribution be exponential or DFR. If the survival distribution is exponential, then the TTT plot should approximate a 45 degree line from the origin. If it is DFR, then the TTT plot should approximate a convex function. In either case, the point of tangency would likely correspond with $t_{(n)}$ provided that there was ample data to give a good approximation.

With small sample sizes, however, it may be difficult to conclude that the data was generated by an exponential or DFR distribution. Given some failure data, we may be able to apply a variety of goodness-of-fit tests to reject the null hypothesis that our data could have come from an exponential (or DFR) distribution. However, if we can not reject the null hypothesis, then we still can not confirm that the data are indeed exponential (or DFR) since it could be that our sample size is insufficient

to detect that the null hypothesis is false. As an alternative goodness-of-fit test, Barlow and Campo (1975) propose counting the number of times the empiric TTT plot crosses the exponential TTT plot. Presumably, exponential observations will result in an empirical TTT plot that crosses the 45 degree line from the origin a large number of times. Bergman (1977b) provides the exact and asymptotic distributions for the number of crossings for exponential data. When using Bergman's graphical method for Age Replacement without knowledge of the underlying distribution, one should be concerned about the probability of estimating a finite Age Replacement interval when, in fact, the distribution is exponential (or DFR).

To address this issue, we simulated 5000 applications of Bergman's method to IFR and DFR lifetime data for various sample sizes. As one would expect, the expected value of the longest lifetime, $E\{T_{(n)}\}$, increases for larger sample sizes from exponential and DFR distributions. As we show in Table 1, Bergman's technique when applied on exponential observations resulted in about a 50 percent probability of incorrectly estimating a <u>finite</u> Age Replacement interval when the cost ratio $C_2/(C_1-C_2)$ equals 1.0. In this table, $P\{t_n\}$ is the probability of making the correct decision not to have preventive replacement. This probability is sensitive to the cost relationship. If $C_2/(C_1-C_2)$ is greater than one (suggesting only a nominal difference between the cost of failure and the cost of

TABLE 1

EXPECTED VALUE OF BERGMAN'S ESTIMATOR, $E\{T_{(J)}\}$, AND PROBABILITY THAT $T_{(J)}$ IS THE LAST FAILURE TIME, $P\{t_n\}$, FOR OBSERVATIONS FROM AN EXPONENTIAL DISTRIBUTION (MEAN = 200)

Sample Size	$E\{T_{(J)}\}$ and $P\{t_n\}$ for $C_2/(C_1-C_2)$ = 1		50		.01	
	$E\{T_{(J)}\}$	$P\{t_n\}$	$E\{T_{(J)}\}$	$P\{t_n\}$	$E\{T_{(J)}\}$	$P\{t_n\}$
10	553.4	.558	585.1	.98	243.5	.11
20	673.2	.520	720.9	.98	220.9	.06
30	742.8	.512	796.0	.98	199.8	.04
40	803.4	.507	855.5	.98	188.4	.03
50	851.4	.513	900.0	.98	182.2	.03
60	894.3	.524	941.8	.98	183.9	.03
70	905.2	.492	959.7	.98	175.4	.02
80	940.7	.507	995.1	.98	177.3	.02
90	972.1	.514	1012.8	.98	172.1	.02
100	978.3	.505	1030.5	.98	165.6	.02

replacement), then there is a higher probability of reaching the correct conclusion of no preventive maintenance. On the other hand, if $C_2/(C_1-C_2)$ is close to the origin (as would be the case with a substantial difference in costs), then there is a large probability of reaching the incorrect conclusion of a finite Age Replacement interval. The reason for this phenomenon is that close to the origin there is a "sample clutter" effect. This occurs when a tangent line (passing through $[-C_2/(C_1-C_2,0])$ strikes a point on the TTT plot associated with a failure time, $t_{(j)}$, less than $t_{(n)}$ where the tangent point may be only slightly above the 45 degree line. Thus, for very large differences between C_1 and C_2, the practitioner ought to view a finite estimate of t* with caution regardless of the sample size. A prudent course of action would be to check for exponentiality (or DFR) with some statistical test and accept the finite estimate only if the data are not exponential (or DFR).

The economic consequences of ignoring this issue might be substantial. If the observed lifetimes are exponential with scale parameter λ, then $t^* = \infty$ and $C(\infty) = \lambda C_1$. However, if Bergman's technique produces a finite estimate, $\hat{t}^*$, of t*, then $C(\hat{t}^*) > C(t^*)$. In Table 2, we show the average relative error in the objective function, $\{C(\hat{t}^*)-C(t^*)\}/C(t^*)$, based on the simulation of 5000 replications of Bergman's technique on exponential observations. Note that, as the cost ratio

TABLE 2

AVERAGE RELATIVE ERROR (RE) IN THE AGE REPLACEMENT OBJECTIVE, $C(\hat{t}^*)$, DUE TO BERGMAN'S TECHNIQUE FOR OBSERVATIONS FROM AN EXPONENTIAL DISTRIBUTION (MEAN = 200)

Sample Size	$C_2/(C_1-C_2) =$ 1	50	.01
	% RE	% RE	% RE
10	6.20	0.35	2.03
20	3.34	0.18	3.12
30	2.45	0.12	4.14
40	1.79	0.09	5.00
50	1.47	0.08	5.65
60	1.24	0.06	6.10
70	1.09	0.05	6.47
80	0.92	0.04	6.75
90	0.88	0.03	7.21
100	0.79	0.03	7.52

decreases, the sample clutter effect aggravates the average relative error in $C(\hat{t}^*)$. We remark that this average relative error is within a nominal 10 percent range for the range of cost ratios shown.

For the Weibull family of distributions, the probability of estimating an infinite replacement age diminishes for increasingly IFR Weibull distributions and increases for more DFR Weibull distributions as we see in Table 3. Of concern, perhaps, is the relatively large probability of incorrectly electing to forego preventive replacement (this is $P\{t_n\}$ in Table 3 for items with a slightly IFR survival distribution). In other words, with Bergman's method, we may erroneously call for the preventive replacement of an item not requiring it and, conversely, in no preventive replacement for an item subject to wearout. The average relative error in $C(\hat{t}^*)$ for these distributions is shown in Table 4.

Bergman's method approximates the failure distribution, $F(t)$, by the California plotting position, i/n, and employs the total time on test statistic to approximate the conditional mean life. Moreover, the true, optimal interval is approximated by one of the observed failure times. The performance of these three approximations is of interest because, as we show below, their synergistic effect on the estimator can be biased. We simulated 5000 replications of Bergman's technique on three IFR Weibull distributions with different shapes

TABLE 3

PROBABILITY THAT BERGMAN'S $t_{(J)}$ IS LAST FAILURE TIME, $P\{t_{(n)}\}$,
FOR OBSERVATIONS FROM A WEIBULL DISTRIBUTION (MEAN = 200)

Sample Size	$P\{t_{(n)}\}$ for $C_2/(C_1-C_2) = 1$ and Weibull Shape Parameter =										
	.5	.6	.7	.8	.9	1.0	1.1	1.2	1.3	1.4	1.5
10	.84	.78	.73	.68	.63	.56	.50	.44	.38	.32	.26
20	.84	.79	.74	.67	.60	.52	.45	.37	.30	.23	.18
30	.86	.81	.74	.67	.59	.51	.43	.35	.27	.20	.14
40	.87	.81	.75	.67	.60	.51	.43	.33	.25	.18	.11
50	.88	.82	.76	.68	.60	.51	.42	.33	.25	.17	.10
60	.88	.83	.76	.68	.60	.52	.42	.33	.24	.16	.10
70	.87	.81	.75	.68	.59	.49	.40	.30	.21	.14	.08
80	.89	.83	.77	.69	.60	.51	.41	.31	.22	.14	.08
90	.89	.83	.77	.69	.60	.51	.40	.31	.22	.13	.07
100	.89	.83	.76	.68	.60	.51	.39	.30	.20	.12	.06
∞	1.00	1.00	1.00	1.00	1.00	1.00	.00	.00	.00	.00	.00

TABLE 4

AVERAGE RELATIVE ERROR (RE) IN THE AGE REPLACEMENT OBJECTIVE, $C(\hat{t}^*)$, DUE TO BERGMAN'S TECHNIQUE FOR OBSERVATIONS FROM VARIOUS WEIBULL DISTRIBUTIONS (MEAN = 200)

Sample Size	% RE for $C_2/(C_1-C_2) = 1$ and Weibull Shape Parameter =										
	.5	.6	.7	.8	.9	1.0	1.1	1.2	1.3	1.4	1.5
10	9.72	9.56	8.39	6.89	6.30	6.2	5.4	5.2	4.6	4.0	3.4
20	3.13	4.75	4.50	4.07	3.43	3.3	3.0	2.6	2.5	2.2	1.9
30	2.48	3.64	3.03	2.65	2.36	2.5	2.3	2.0	1.9	1.6	1.5
40	0.85	2.41	2.20	2.00	1.67	1.8	1.7	1.6	1.5	1.2	1.2
50	0.65	2.05	1.82	1.65	1.37	1.5	1.5	1.3	1.4	1.1	1.0
60	0.20	1.84	1.66	1.54	1.15	1.2	1.3	1.2	1.1	1.0	0.9
70	0.09	1.58	1.37	1.19	0.87	1.1	1.0	0.9	0.9	0.9	0.8
80	0.00	1.20	1.20	1.00	0.73	0.9	0.9	0.8	0.8	0.7	0.7
90	0.00	1.10	1.10	0.97	0.72	0.9	0.9	0.8	0.9	0.7	0.7
100	0.00	1.05	0.88	0.80	0.54	0.8	0.7	0.7	0.8	0.7	0.6

which result in a finite t* for any cost ratio, $0 < -C_2/(C_1-C_2) < \infty$. These different shaped distributions are convex ordered, and hence, a large shape parameter distribution is said to be more IFR than a small shape parameter distribution. Accordingly, these three distributions, which we consider, are nominally, moderately, and highly IFR, respectively. For increasingly IFR distributions, there is a decreasing probability of estimating t* as infinity (see $P\{t_{(n)}\}$ in Table 5). We observe, from Table 5, that the expected value of Bergman's estimator, $E\{\hat{T}^*\}$, has a positive bias for moderate to highly IFR Weibull distributions and a negative bias for the nominally IFR distribution.

As an alternative to estimating t* with a graphical technique that approximates the conditional mean life using the total time on test statistic, we could replace this statistic with the survivor time on test statistic. The resultant estimate has a negative bias for the low to moderately IFR distributions and a positive bias for the highly IFR distribution as we show in Table 6. A combination of these two estimators, then, may result in less bias than either estimator alone if the failure distribution is moderately IFR.

Summary

Graphical solutions to the Age Replacement problem offer an attractive alternative to analytical solutions

TABLE 5

EXPECTED VALUE OF BERGMAN'S ESTIMATOR, $E\{\hat{T}^*\}$; PROBABILITY THAT $t_{(J)}$ IS THE LAST FAILURE TIME, $P\{t_{(n)}\}$; AND AVERAGE % RELATIVE ERROR (% RE) IN $C(t^*)$ FOR OBSERVATIONS FROM VARIOUS WEIBULL DISTRIBUTIONS (MEAN = 200) USING TTT STATISTIC

Sample Size	for $C_2/(C_1-C_2)$ = 1.0 and Weibull Shape of								
	1.5			2.0			3.0		
	$E\{T^*\}$	$P\{t_n\}$	% RE	$E\{T^*\}$	$P\{t_n\}$	% RE	$E\{T^*\}$	$P\{t_n\}$	% RE
10	358.2	.26	3.45	261.1	.08	6.17	194.6	.00	2.69
20	395.1	.18	1.93	261.5	.02	0.61	189.5	.00	2.00
30	409.9	.14	1.48	257.8	.01	0.57	187.2	.00	1.65
40	420.9	.11	1.19	256.4	.00	0.84	187.3	.00	1.37
50	430.7	.10	1.02	254.7	.00	0.76	186.3	.00	1.18
60	438.3	.10	0.86	255.3	.00	0.90	185.4	.00	1.05
70	441.8	.08	0.76	254.6	.00	0.75	184.6	.00	0.93
80	449.0	.08	0.69	252.8	.00	0.79	184.4	.00	0.87
90	448.4	.07	0.65	252.2	.00	0.72	184.2	.00	0.81
100	449.5	.06	0.58	253.1	.00	0.72	183.9	.00	0.75
∞	481.0	.00	-	249.9	.00	-	182.7	.00	-

TABLE 6

EXPECTED VALUE OF BERGMAN'S ESTIMATOR, $E\{\hat{T}^*\}$; PROBABILITY THAT $t_{(J)}$ IS THE LAST FAILURE TIME, $P\{t_{(n)}\}$; AND AVERAGE % RELATIVE ERROR (% RE) IN $C(t^*)$ FOR OBSERVATIONS FROM VARIOUS WEIBULL DISTRIBUTIONS (MEAN = 200) USING STT STATISTIC

Sample Size	for $C_2/(C_1-C_2)$ = 1.0 and Weibull Shape of 1.5		2.0		3.0	
	E{T*}	% RE	E{T*}	% RE	E{T*}	% RE
10	266.2	6.52	221.5	3.35	185.9	2.77
20	309.6	3.26	235.1	1.93	185.6	1.97
30	328.5	2.35	239.0	1.49	185.2	1.60
40	342.2	1.86	242.4	1.19	185.2	1.33
50	352.5	1.55	243.0	1.04	185.1	1.15
60	363.2	1.27	244.4	0.93	184.3	1.03
70	372.6	1.11	245.5	0.86	183.6	0.91
80	376.9	1.00	245.2	0.78	183.7	0.86
90	380.7	0.93	245.3	0.73	183.5	0.80
100	386.6	0.81	246.8	0.70	183.3	0.74
∞	481.0	-	249.9	-	182.7	-

and their inherent uncertainty of the failure distribution and the complexity of search routines to solve for the minimizer of either equations (1.2) or (2.1). Bergman (1977a) has proposed a simple graphical method involving the total time on test plot and an empirical failure distribution. The sampling issues of this approach are of interest when we consider statistical inferences from a set of observed failures. We find that Bergman's method may incorrectly lead to finite estimates of the optimal Age Replacement interval if there are substantial differences between the costs of failure and replacement and if the (unknown) parent failure distribution is exponential or DFR. The economic consequences, however, in terms of the percent relative error are nominal. Nonetheless, Bergman's method may accurately estimate optimal Age Replacement intervals for many IFR distributions with samples as small as 20 observations.

CHAPTER III

BLOCK REPLACEMENT

Block Replacement is a maintenance strategy which does not require knowledge of an item's age. Such information may be expensive or impossible to obtain. Rather, with Block Replacement, we replace or renew to good-as-new condition at failure and at periodic intervals of time regardless of when the last failure occurred. Barlow and Proschan (1965; 1975) discuss Block Replacement and remark that, when compared to Age Replacement, Block Replacement results in more removals regardless of the survival distribution and in fewer failures when the survival distribution is IFR. Although more useful life is discarded under Block Replacement than Age Replacement, there is no necessity to record age.

Presuming a failure cost C_1 and a preventive replacement cost C_2, Barlow and Proschan (1965) formulate a long run, average cost per unit time function, B(t), as

$$B(t) = \frac{C_1 M(t) + C_2}{t} \tag{3.1}$$

They show the first order necessary condition to be

$$m(t)t - M(t) = \frac{C_2}{C_1} \tag{3.2}$$

where m(t) is the derivative of the renewal function, M(t), which is the sum of the k-fold convolution of the failure distribution with itself for $k=1, \ldots, \infty$. As we explain in Chapter I, the renewal function gives the expected number of failures in the interval (0,t]. Barlow and Proschan reason that B(t) approaches infinity as t approaches zero and conclude that B(t) has a minimizer, t*, in the interval $(0,\infty]$ where $t^*=\infty$ is interpreted as replacement at failure only. As we also explain in Chapter I, a graphical clarity can be obtained by considering the standardized objective

$$\tilde{B}(t) = \frac{B(t)}{C_1} = \frac{M(t) + \frac{C_2}{C_1}}{t} . \tag{3.3}$$

The renewal function can be expressed in terms of the fundamental renewal equation

$$M(t) = F(t) + \int_0^t M(t-x)\,dF(x) \tag{3.4}$$

which is often solved via the Laplace-Stieltjes transform. There are several basic results for selected processes which we use in later examples. Karlin and Taylor (1975)

characterize a Poisson process with parameter λ as a renewal process and note that $M(t) = \lambda t$, a linear function. They also give a closed form solution for the renewal function of an IFR gamma distribution with shape parameter of two and scale parameter beta as

$$M(t) = \frac{t}{2\beta} - \frac{1-\exp\left(\frac{-2t}{\beta}\right)}{4} \tag{3.5}$$

This renewal function is a convex function in t. As for the DFR class of distributions, Brown (1980) proves that their renewal functions are concave. We make use of these geometric properties in a graphical solution to Block Replacement.

Earlier in Figure 7, we depicted a geometric solution to the Block Replacement model by plotting $y = M(t)$ versus t. If the lifetime distribution is exponential, then the renewal function, $M(t)$, is a linear function in t so that there is no finite tangent point as long as $C_2/C_1 > 0$. Hence, the optimal Block Replacement policy for exponential lifetimes is to replace at failure only. However, an IFR gamma distribution, with shape parameter two and scale parameter beta, has a convex renewal function in t. Barlow and Proschan show that this renewal function results in a finite optimal Block Replacement interval so long as C_1 and C_2 satisfy the inequality $C_2/C_1 < .25$. DFR distributions, on the other hand, have concave renewal functions and, hence, optimal Block

Replacement intervals of infinity. Notwithstanding these geometric properties, an analytic or numerical solution to Block Replacement requires specification of the renewal function or at least the lifetime distribution. As we outlined in Chapter I, we can address our uncertainty of the renewal function by approximating it by an empirical renewal function developed from a superposition process as in Figure 8.

The superimposed number of failures, $\sum_{i=1}^{n} N_i(t)$ form the series $\{Z_{(1)}, Z_{(2)}, \ldots, Z_{(m)}\}$ where the last superimposed failure time, $Z_{(m)}$, is determined by some arbitrary life test termination time t, i.e. $Z_{(m)} \leq t$. Then, by the law of large numbers,

$$\lim_{n \to \infty} \frac{\sum_{i=1}^{n} N_i(t)}{n} = M(t) .$$

For n finite, we can approximate M(t) as

$$M(t) \simeq M_n(t) = \frac{m}{n} .$$

The number of events, m, in this superposition process is a function of t. There are two methods of determining t which are akin to the notion of life test censoring. Either the life test is stopped at a specified time, t_0, or it is terminated after a particular number of failures, say "r," have occurred at each component position. In the first case, both the number of superimposed failures

(possibly zero) and their failure times are random variables. In the latter case, only the superimposed failure times are random variables since the number of failures is fixed (at some positive quantity). We choose to terminate after r failures occur in each component position, thereby guaranteeing a quantity of m superimposed failures. Here, m will be equal to the product of the number of component positions, n, and the number of failures at each position, r.

The order in which these r lifetimes will occur in any component position is random. For any given set of r lifetimes, there are r! possible sample paths representing the permutations in which these lifetimes could occur. We can make full use of the information in the n sets of r lifetimes by sampling within each set to obtain more sets of r lifetimes with a different order of occurrence. We discuss this issue later as a data augmentation technique.

Graphical Solution to Block Replacement

Following an argument similar to that of the graphical approach to Age Replacement, we can solve an approximation to equation (3.3) by direct numeration methods. Given a set of superimposed failure times, $\{z_{(1)}, \ldots, z_{(m)}\}$, we estimate the minimizer, t^*, of $\tilde{B}(t)$ by one of the summed failure times, $\hat{t}^* = z_{(j)}$, such that

$$\tilde{B}(t^*) \simeq \tilde{B}(z_{(j)}) = \frac{M(z_{(j)}) + \frac{C_2}{C_1}}{z_{(j)}} \tag{3.6}$$

Since $M_n(z_{(j)}) = j/n$, we can further approximate $\tilde{B}(z_{(j)})$ by this empiric renewal function such that

$$\tilde{B}(z_{(j)}) \simeq \frac{\frac{j}{n} + \frac{C_2}{C_1}}{z_{(j)}} \tag{3.7}$$

Our strategy is to initiate a life test with n component positions and run the test until r failures have occurred in each component position. We then evaluate equation (3.7) for each summed failure time, $z_{(i)}$, and identify that $z_{(j)} = \hat{t}^*$ which minimizes $\hat{B}(z_{(i)})$. *The accuracy of* our estimate will, in part, be a function of how many component positions we use in the life test as well as the number of failures at each position. If $\hat{t}^*$ happens to be $z_{(m)}$, then two possibilities arise; either the true optimal Block Replacement interval is indeed infinity or the optimal interval, t^*, is greater than $z_{(m)}$ but still finite. To resolve this question, a practitioner should extend the termination point (i.e. increase r) by some reasonable amount with more life testing in each component position.

We investigated the question of accuracy by simulating 1000 applications of the above methodology for IFR and DFR lifetime data with various numbers of component

positions. In Table 7, we show the expected value of our estimator, $E\{\hat{T}^*\}$, for increasing numbers of component positions, n, when the survival distribution is an IFR gamma distribution (shape parameter two, scale parameter one). Recall that for cost ratios (C_2/C_1) less than .25, there should be a finite Block Replacement interval. As we see in Table 7, the expected value of our estimator, $E\{\hat{T}^*\}$, overestimates t* for large differences between C_1 and C_2, say $C_2/C_1 < .04$, but converges toward t* as the number of positions increase. We note that the probability of $\hat{t}^*$ being the last observed superimposed failure, $P\{\hat{t}^* = z_{(m)}\}$ (which we denote as $P\{z_m\}$), is small as long as the cost ratio is small. However, as the cost ratio increases there is a greater likelihood that $\hat{t}^* = z_{(m)}$. Thus, for cost ratios as small as ten to one, (which should result in a finite $\hat{t}^*$ for this distribution) there is a substantial probability of incorrectly estimating the optimal Block Replacement interval as infinity. This accounts for the lack of convergence of $E\{\hat{T}^*\}$, as shown in Table 7, when $C_2/C_1 = .10$. If $\hat{t}^* = \infty$ (as would be the case if the point of tangency corresponds to $z_{(m)}$), then by the elementary renewal theorem, $B(\hat{t}^* = \infty) = C_1/\mu$ where μ is the expected value of a lifetime generated by the underlying distribution. For finite $\hat{t}^*$, we can compute $B(\hat{t}^*)$ provided that we have a closed form solution for $M(\hat{t}^*)$ as in equation (3.5). Accordingly, in Table 8, we show the average relative error, $\{[B(\hat{t}^*)-B(t^*)]/B(t^*)\}$,

TABLE 7

ESTIMATED OPTIMAL BLOCK REPLACEMENT INTERVAL, $E\{Z_J\}$, AND PROBABILITY OF LAST FAILURE TIME, $P\{z_m\}$, FOR A GAMMA DISTRIBUTION (SHAPE = 2, SCALE = 1)

10 Obs in # of Positions	Cost Ratio, C_2/C_1							
	.10		.04		.02		.01	
	$E\{Z_J\}$	$P\{z_m\}$	$E\{Z_J\}$	$P\{z_m\}$	$E\{Z_J\}$	$P\{z_m\}$	$E\{Z_J\}$	$P\{z_m\}$
10	10.4	.292	5.24	.130	4.04	.093	3.41	.076
20	12.8	.379	4.59	.122	2.67	.063	2.21	.050
30	13.9	.419	3.51	.090	1.68	.036	1.14	.021
40	16.7	.506	3.63	.095	1.31	.025	0.78	.011
50	17.9	.538	2.91	.071	1.02	.018	0.57	.006
60	20.2	.597	3.34	.082	0.93	.015	0.57	.007
70	21.9	.643	3.41	.086	1.18	.023	0.49	.005
80	23.6	.692	3.52	.089	0.79	.013	0.43	.004
90	24.6	.723	3.20	.078	0.74	.011	0.37	.002
100	25.5	.742	3.33	.079	0.86	.014	0.35	.002
∞	.688	.000	.356	.000	.233	.000	.155	.000

TABLE 8

AVERAGE RELATIVE ERROR (RE) IN THE BLOCK REPLACEMENT OBJECTIVE, $B(t^*)$, DUE TO THE ESTIMATOR, $\hat{T}^*$, FOR A GAMMA DISTRIBUTION (SHAPE = 2, SCALE = 1)

	Cost Ratio, C_2/C_1			
	.10	.04	.02	.01
10 Obs in # of Positions	% RE in $B(t^*)=3.73$	% RE in $B(t^*)=6.37$	% RE in $B(t^*)=9.31$	% RE in $B(t^*)=13.46$
10	15.80	36.45	66.37	113.79
20	16.06	25.51	41.57	72.49
30	16.34	18.39	27.96	49.30
40	18.61	17.29	21.77	36.80
50	19.61	13.20	16.66	29.70
60	21.42	13.88	14.28	24.69
70	22.98	13.68	14.55	22.24
80	24.28	13.56	11.32	18.39
90	25.19	12.34	10.46	17.78
100	25.76	11.87	10.01	15.24

in the objective function due to our estimator $\hat{T}^*$ when the failure distribution is gamma with shape parameter equal two and scale equal one. These results lead to the intuitive conclusion that when there is a large cost difference between C_1 and C_2, we should employ this solution technique with a large number of component positions.

Renewal function plots of DFR distributions are concave. The exponential distribution, as a special case DFR distribution, has a renewal function that is linear in t. Theoretically, as long as the cost ratio C_2/C_1 is finite, the slope b* of the tangent $y = b^*t - C_2/C_1$ (shown in Figure 7) will be zero and $t^* = \infty$. However, with an empirical renewal function, we are approximating this linear (or concave) renewal function with a piece-wise linear function which may have a finite point of tangency. In fact, as we show in Table 9, smaller cost ratios (i.e. closer to the origin), result in decreasing probability of correctly estimating an infinite Block Replacement interval. However, the consequences of this statistical error appear to be minimal as evidenced by the low relative error in $B(\hat{t}^*)$ shown in Table 9.

Data Augmentation Technique for Block Replacement

The empirical, renewal function estimator discussed above may require failure data from a large number of component positions to estimate adequately the optimal Block Replacement interval, t*, of an IFR class

TABLE 9

PROBABILITY OF LAST FAILURE TIME, $P\{z_m\}$, AND AVERAGE RELATIVE ERROR (RE) IN THE BLOCK REPLACEMENT OBJECTIVE, B(t*), DUE TO THE ESTIMATOR $\hat{T}^*$ FOR AN EXPONENTIAL DISTRIBUTION (MEAN = 1)

10 Obs in # of Positions	Cost Ratio, C_2/C_1							
	.1		.04		.02		.01	
	$P\{z_m\}$	% RE	$P\{z_m\}$	% RE	$P\{z_m\}$	% RE	$P\{z_m\}$	% RE
10	.677	2.77	.602	2.22	.561	1.51	.528	0.92
20	.887	0.94	.786	2.07	.725	1.79	.667	1.32
30	.934	0.17	.882	1.00	.814	1.59	.759	1.36
40	.953	0.05	.918	0.69	.854	1.28	.773	1.52
50	.968	0.02	.948	0.40	.893	1.27	.820	1.56
60	.967	0.02	.959	0.15	.929	0.65	.868	1.07
70	.974	0.01	.970	0.12	.939	0.73	.883	1.13
80	.977	0.01	.972	0.10	.959	0.35	.913	0.91
90	.978	0.01	.975	0.10	.961	0.29	.917	0.83
100	.983	0.01	.982	0.02	.971	0.25	.940	0.71

distribution. For small numbers of component positions, the estimator may overestimate t*. The impact of scant data might be diminished, however, by "creating" more data with a sampling scheme. In this scheme, the r lifetimes in a percentage of the component positions are randomly reordered and treated as additional component position failure data. These additional data, of course, are not independent of the original failure data. However, given a large number of lifetimes in each position, a randomly selected sequence of failure times may result in another sample path which is weakly dependent on the original sample path. Hence, our estimating technique might be robust insofar as the requirement for independent sample paths is concerned.

To illustrate this sampling idea via Monte Carlo simulation, we used a sequential sampling plan where we randomly sampled the r lifetimes in alternate component positions. We drew r observations with replacement and recorded them, in the order drawn, as failures in pseudo-positions. Then, we superimposed the failure times in both the actual positions and the pseudo-positions to form the series $\{y_{(1)} \leq \cdots \leq y_{(m)}\}$ where m now equals 1.5 nr. With this sampling plan, we artificially increased the number of superimposed data points by 50 percent and (we hope) improved our optimal Block Replacement estimator. In Table 10, we display the results of 1000 iterations of this data augmentation technique to produce an expected

TABLE 10

A COMPARISON OF E{T*} VERSUS E{T'} AND AVERAGE % RELATIVE ERROR (% RE) IN B{t'} FOR A GAMMA (SHAPE = 2, SCALE = 1) DISTRIBUTION

10 Obs in # of Positions	Cost Ratio, C_2/C_1								
	.04			.02			.01		
	E{T*}	E{T'}	% RE	E{T*}	E{T'}	% RE	E{T*}	E{T'}	% RE
10	5.24	6.13	30.6	4.04	4.34	53.1	3.41	3.37	90.5
20	4.59	5.92	23.5	2.67	3.08	32.8	2.21	1.95	52.1
30	3.51	6.02	20.7	1.68	2.36	23.5	1.14	1.26	36.1
40	3.63	6.70	20.6	1.31	2.39	20.0	0.78	1.08	29.1
50	2.91	7.57	21.6	1.02	2.27	17.2	0.57	0.76	22.7
60	3.34	7.82	21.3	0.93	2.00	14.9	0.57	0.70	19.0
70	3.41	8.54	22.6	1.18	2.20	15.0	0.49	1.08	19.7
80	3.52	9.09	23.3	0.79	1.61	11.3	0.43	0.52	13.8
90	3.20	9.78	24.5	0.74	1.74	12.0	0.37	0.36	13.3
100	3.33	8.72	21.9	0.86	1.57	10.9	0.35	0.46	11.4
∞	.356	.356	-	.233	.233	-	.155	.155	-

value of the estimate $\hat{t}' = y_{(j)}$ compared to the expected value of the previous estimate $\hat{t}^*$ for a gamma (shape 2, scale 1) distribution with various cost ratios resulting in a finite t^*.

At first glance, this data augmentation technique does not appear to result in an improved estimator except when the cost ratio is very small (see $C_2/C_1 = .01$ in Table 10) and there are few numbers of component positions. However, when the relative error in $B\{\hat{t}'\}$ is compared to the relative error in $B\{\hat{t}^*\}$, shown in Table 10, we note that the estimate $\hat{t}'$ results, on average, in smaller relative error. This advantage diminishes as the number of component positions, n, increase, and for n greater than 30 (in Table 10), $\hat{t}^*$ would be preferred to $\hat{t}'$. Notwithstanding this disadvantage, our data augmentation technique may be useful when there are scant data.

Summary

A graphical solution to Block Replacement can be developed by using an empiric estimator of the renewal function based on a superposition process concept. The idea is to superimpose r lifetimes from each of n component positions as $\{z_{(1)} \leq \dots \leq z_{(rn)}\}$ and estimate the expected number of renewals in the interval $[0, z_{(i)}]$ as i/n. The graphical technique, then, identifies that $z_{(j)}$ which corresponds to the point of tangency on a plot of the empiric, renewal function for a line drawn through the

point $[0,-C_2/C_1]$. This $z_{(j)}$ is an estimator of the optimal Block Replacement interval, t*, which minimizes long run, average cost per unit time. The accuracy of this estimator can be enhanced by increasing the number of component positions. This is especially important when the difference between the cost of replacement and the cost of failure is great.

CHAPTER IV

BLIND REPLACEMENT

"Inspection policies" are a class of maintenance models characterized by two kinds of uncertainty. For this class, not only are we uncertain of an item's lifetime, which can be expressed in terms of a probability distribution, but we are also uncertain of the item's condition (good or failed). Savage (1956) and others have referred to this class as that of "preparedness models" pertaining to items in storage that may be called into service during some random emergency. We choose to broaden this definition to include in-service items, with an emphasis on the matter of item condition and its uncertainty. To determine condition, we generally inspect and then replace (or repair) *a defective item upon discovery.* For some items, the cost of inspection exceeds the cost of replacement. For example, automotive engine crankcase lubricant is commonly replaced at periodic intervals rather than inspected for condition and replaced as necessary. In this case, we presume the cost of inspection (a laboratory analysis) greatly exceeds the cost of replacement. For other items such as explosive devices, an inspection involves consumption of the item, and hence

condition can never be determined, for to do so would terminate lifetime with certainty. It is this set of circumstances that Radner and Jorgenson (1962) describe in their model for the replacement of a single-part equipment with an arbitrary lifetime distribution. We will refer to this model as Blind Replacement.

In Blind Replacement, Radner and Jorgenson presume that each replacement consumes K units of time. They formulate an objective as a function of the replacement interval, t, which maximizes long run, average "time good" per unit time so that

$$G(t) = \frac{\int_0^t R(x)\,dx}{t+K} . \tag{4.1}$$

Radner and Jorgenson show that equation (4.1) is a concave function in t and remark that first order optimality conditions are sufficient to produce a unique maximizer, t*, which satisfies

$$(t+K)\ R(t) = \int_0^t R(x)\,dx . \tag{4.2}$$

Illustrating the use of equation (4.2) for exponential lifetimes, they develop a transcendental equation for t* and observe that "the optimal time to replacement increases with an increase in the time required for replacement." Moreover, they remark that t* increases as

the mean exponential lifetime increases. Regardless of the lifetime distribution, we note that to use either equation (4.1) or (4.2), we must specify the survival distribution, $R(x)$, as well as the replacement time, K. In practice, we may be uncertain of either and, therefore, may find empirical techniques similar to those of Bergman to be useful.

Graphical Solutions for Blind Replacement

There are two straightforward geometric solution techniques for Blind Replacement which use the same approximations that Bergman uses in Age Replacement and which, as a consequence, may be subject to bias. Given a set of complete, life test data, $\{t_{(1)} \leq \cdots \leq t_{(n)}\}$, we can evaluate either equation (4.1) or (4.2) for each $t_{(i)}$, $(i = 1, \ldots, n)$ and estimate t^* by the maximizer, $t_{(j)}$, from the set of failure times. Using equation (4.1), we can numerically evaluate the objective function at each $t_{(i)}$ and identify the $t_{(j)}$ which maximizes $G(t_{(i)})$ as an estimate, $\hat{t}^*$, of the optimal interval. We refer to this as a "direct numeration" method. Another method evaluates equation (4.2) for each $t_{(i)}$, and it identifies that $t_{(j)}$ which most closely satisfies the quality as the "first order" estimate, $\hat{t}^@$, of the optimal Blind Replacement interval.

Direct Numeration Solutions

The graphical argument for the direct numeration solution was portrayed earlier in Figure 10. Our approach involves estimating t^* by one of the ordered lifetimes and approximating the conditional mean life, $\mu(t_{(i)}) = \int_0^{t_{(i)}} R(x)dx$, by the total time on test statistic, $T(t_{(i)})$, divided by n. Theoretically, for infinite tailed distributions, a plot of the conditional mean life should increase asymptotically to the mean life. Thus, (from the geometry in Figure 10) as long as the replacement time, K, is finite, the optimal Blind Replacement interval will be finite, i.e. in Blind Replacement, we always replace eventually regardless of the underlying distribution. So, if this graphical technique results in an estimate $t_{(j)} = t_{(n)}$ of t^*, then we replace at $t_{(n)}$ rather than forego preventive replacement as we would in Age Replacement.

To evaluate possible bias in our estimator, we simulated 1000 iterations of this graphical solution technique for various sample sizes from selected IFR and DFR Weibull distributions and a uniform distribution. Our results indicate that for a sufficiently small replacement time, K, $\hat{t}_{(j)} = t^*$ will converge to t^* from above as the sample size increases, i.e., $t^* \leq \hat{t}^*$. For a very large K, however, convergence may be from below. The percent relative error in the objective function, $G(\hat{t}^*)$, due to the biased estimate is provided below. For the range of cases

considered, this percent relative error is within 10 percent.

For an exponential distribution, we can note in Table 11, convergence from above for replacement times as large as three times the mean lifetime. The rapidity of convergence appears to be a function of the replacement time, with convergence to within 5 percent of the actual optimal interval for sample sizes as small as 30. As indicated in Tables 12 and 13, convergence for IFR distributions may be more rapid than for DFR distributions with the same mean lifetime. For these distributions there is a concave ordering such that conditional mean life plot of IFR distributions are more concave than conditional mean life plots of DFR distributions. Thus, the geometry of the direct numeration method leads to rapid convergence for increasingly IFR distributions. Results for the uniform distribution, shown in Table 14, also demonstrate convergence from above for small K and convergence from below for large K.

Of interest is the possibility of mitigating the bias in this direct numeration technique. For a small sample (say n less than 20) and a small-to-moderate replacement time, K, we might presume that our estimate might be greater than t^*. Of course, if we collect more data, our estimate, $\hat{t}^*$, of the optimal interval would converge, i.e. there would be less bias in the estimate. However, without any additional data collection, we could

TABLE 11

EXPECTED VALUE OF THE OPTIMAL BLIND REPLACEMENT ESTIMATOR, E{T*}, AND THE AVERAGE RELATIVE ERROR (RE) IN G(t*) FOR AN EXPONENTIAL DISTRIBUTION (MEAN = 200) WITH A DIRECT NUMERATION TECHNIQUE

Sample Size	for Replacement Time, K =									
	10		400		800		1200		5000	
	E{T*}	% RE	E{T*}	% RE	E{T*}	% RE	E{T*}	% RE	E{T*}	% RE
10	67.2	1.92	309.7	3.01	393.9	3.48	457.8	3.20	585.1	4.54
20	63.9	0.98	306.8	1.57	393.6	1.67	452.3	1.70	717.8	2.02
30	62.7	0.65	304.6	1.00	391.4	1.11	447.9	1.18	684.4	1.24
40	61.9	0.47	303.3	0.77	390.7	0.86	447.2	0.88	662.0	1.15
50	61.7	0.38	303.0	0.59	389.0	0.65	446.5	0.71	689.6	0.84
60	61.3	0.31	303.1	0.53	389.2	0.57	446.6	0.60	676.2	0.72
70	61.1	0.29	302.2	0.42	388.3	0.46	445.3	0.51	671.2	0.62
80	61.1	0.24	301.8	0.37	388.2	0.42	445.0	0.45	682.1	0.50
90	60.8	0.21	301.5	0.32	387.1	0.40	443.9	0.42	675.3	0.48
100	60.6	0.19	302.2	0.30	388.8	0.34	445.4	0.36	673.2	0.42
∞	60.1	-	301.1	-	387.4	-	444.3	-	676.1	-

TABLE 12

EXPECTED VALUE OF THE OPTIMAL BLIND REPLACEMENT ESTIMATOR, E{T*}, AND THE AVERAGE RELATIVE ERROR (RE) IN G(t*) FOR AN IFR WEIBULL DISTRIBUTION (SHAPE = 2, MEAN = 200) USING THE TOTAL TIME ON TEST STATISTIC

Sample Size	for Replacement Time, K = 10		50		200		400		500	
	E{T*}	% RE	E{T*}	% RE	E{T*}	% RE	E{T*}	% RE	E{T*}	% RE
10	96.3	1.26	149.8	1.27	217.3	1.34	257.6	1.54	302.2	1.55
20	93.4	0.58	150.0	0.66	218.3	0.72	258.3	0.74	299.6	0.78
30	91.7	0.43	148.8	0.44	217.8	0.44	257.5	0.47	299.0	0.49
40	91.6	0.30	148.3	0.34	217.6	0.35	257.4	0.38	299.8	0.39
50	90.9	0.27	147.8	0.26	217.4	0.27	257.7	0.29	300.0	0.28
60	90.8	0.21	147.9	0.22	217.4	0.34	257.6	0.25	299.5	0.24
70	90.6	0.18	147.8	0.20	217.4	0.19	257.9	0.21	300.2	0.21
80	90.4	0.16	147.8	0.16	217.3	0.18	257.5	0.18	299.4	0.19
90	90.1	0.14	147.6	0.15	217.3	0.15	258.0	0.15	300.6	0.17
100	90.1	0.13	147.8	0.13	217.4	0.16	257.7	0.15	299.9	0.14
∞	89.3	-	147.2	-	217.2	-	257.5	-	300.0	-

TABLE 13

EXPECTED VALUE OF THE OPTIMAL BLIND REPLACEMENT ESTIMATOR, E{T*}, AND THE AVERAGE RELATIVE ERROR (RE) IN G(t*) FOR DFR WEIBULL DISTRIBUTION (SHAPE = .5, MEAN = 200) WITH A DIRECT NUMERATION TECHNIQUE

Sample Size	for Replacement Time, K = 10		50		200		400		800	
	E{T*}	% RE	E{T*}	% RE	E{T*}	% RE	E{T*}	% RE	E{T*}	% RE
10	51.8	3.38	135.1	4.14	301.9	5.83	446.7	6.49	681.4	7.75
20	46.5	1.55	123.2	1.98	282.2	2.93	416.7	3.26	606.4	4.06
30	44.9	1.07	119.6	1.27	269.8	1.91	399.5	2.28	585.9	2.77
40	44.0	0.82	117.8	1.04	267.7	1.46	395.1	1.74	579.0	2.02
50	43.6	0.61	117.4	0.82	266.9	1.09	395.0	1.26	574.4	1.56
60	43.2	0.51	116.8	0.66	263.9	0.99	390.9	1.06	567.5	1.35
70	43.1	0.46	116.3	0.53	264.2	0.80	392.3	0.91	573.4	1.15
80	42.9	0.39	115.6	0.50	262.4	0.68	387.4	0.81	566.8	1.02
90	42.7	0.35	115.9	0.44	264.6	0.57	391.1	0.80	570.0	0.95
100	42.8	0.33	115.4	0.43	262.3	0.52	387.9	0.64	565.2	0.78
∞	41.9	-	113.9	-	258.9	-	383.1	-	557.9	-

TABLE 14

EXPECTED VALUE OF THE OPTIMAL BLIND REPLACEMENT ESTIMATOR, $E\{T^*\}$, AND THE AVERAGE RELATIVE ERROR (RE) IN $G(t^*)$ FOR A UNIFORM [0,400] DISTRIBUTION USING THE TOTAL TIME ON TEST STATISTIC

Sample Size	for Replacement Time, K =									
	10		50		200		400		800	
	E{T*}	% RE	E{T*}	% RE	E{T*}	% RE	E{T*}	% RE	E{T*}	% RE
10	90.9	1.87	162.9	1.71	247.9	1.45	288.9	1.42	324.4	1.22
20	85.3	0.91	159.5	0.84	247.4	0.75	290.5	0.65	327.0	0.52
30	83.1	0.61	158.2	0.58	247.2	0.52	290.8	0.49	327.9	0.40
40	82.5	0.44	157.6	0.44	247.7	0.40	291.9	0.34	329.5	0.27
50	81.8	0.34	157.0	0.34	246.7	0.32	292.0	0.28	329.7	0.24
60	81.4	0.29	156.7	0.29	247.1	0.26	292.0	0.23	329.7	0.17
70	81.5	0.23	157.2	0.24	246.9	0.22	292.2	0.21	329.9	0.17
80	81.3	0.22	156.7	0.21	247.1	0.21	292.1	0.19	330.2	0.15
90	81.2	0.18	156.9	0.19	247.4	0.18	292.2	0.15	330.3	0.12
100	80.9	0.17	156.9	0.19	247.2	0.16	292.6	0.14	330.5	0.12
∞	80.0	-	156.2	-	247.2	-	292.8	-	331.4	-

reaccomplish our graphical technique using the survivor time on test statistic divided by n as an approximation of the conditional mean life in lieu of the total time on test statistic.

Direct Numeration with Survivor Time on Test Statistic

We may be able to improve on the estimate of t^* by using the survivor time on test statistic, discussed in Chapter I, in conjunction with the total time on test statistic. Our strategy is to approximate the conditional mean life by the survivor time on test statistic divided by n and identify as $\hat{t}^{\#}$ the point of tangency corresponding to the tangent line from the point $(-K,0)$. One naive approach is to estimate t^* as the average between $\hat{t}^{\#}$ and $\hat{t}^*$. For some small replacement times, $\hat{t}^{\#}$ may not converge to the optimal replacement interval from below because of the geometry of the direct numeration method (see Table 15). For large n, of course, there is no appreciable difference between $\hat{t}^{\#}$ and $\hat{t}^*$.

The composite estimator $\hat{t}^C = (\hat{t}^* + \hat{t}^{\#})/2$ can produce a smaller relative error than the estimator $\hat{t}^*$ as we show in Table 16 for the exponential distribution. This advantage of the composite estimator, $\hat{t}^C$, increases as the replacement time, K, increases; however, it diminishes as the sample size increases because of the convergence of $\hat{t}^*$ and $\hat{t}^C$. A prudent strategy, then, would be to use such

TABLE 15

EXPECTED VALUE OF THE OPTIMAL BLIND REPLACEMENT ESTIMATOR, E{T*}, FOR OBSERVATIONS FROM AN EXPONENTIAL DISTRIBUTION (MEAN = 200) WITH DIRECT NUMERATION USING THE SURVIVOR TIME ON TEST STATISTIC

Sample Size	E{T*} for Replacement Time, K =				
	10	50	100	200	400
10	63.4	122.6	160.5	208.4	261.4
20	61.8	124.4	167.0	219.4	281.6
30	61.4	125.2	168.0	222.3	287.8
40	60.9	126.2	169.8	225.1	291.2
50	60.9	126.7	170.1	226.0	292.9
60	60.7	126.0	170.5	226.6	294.7
70	60.5	125.9	169.9	226.2	294.8
80	60.5	125.9	170.5	226.4	295.5
90	60.5	126.1	170.3	226.8	296.0
100	60.2	126.2	170.7	227.1	297.1
∞	60.1	126.5	171.6	229.3	301.1

TABLE 16

AVERAGE PERCENT RELATIVE ERROR (%RE) IN $\underset{\sim}{G}(t^*)$ DUE TO ESTIMATOR $\hat{t}^*$
VERSUS A COMBINATION ESTIMATOR $\hat{t}^c = (t^* + t^\#)/2$
FOR AN EXPONENTIAL DISTRIBUTION (MEAN = 200)
WITH A DIRECT NUMERATION TECHNIQUE

Sample Size	for Replacement Time, K = 10		50		100		200	
	t^*%RE	t^c%RE	t^*%RE	t^c%RE	t^*%RE	t^c%RE	t^*%RE	t^c%RE
10	1.92	1.89	2.27	2.14	2.40	2.26	2.64	2.51
20	0.98	0.94	1.08	1.06	1.13	1.12	1.35	1.32
30	0.65	0.63	0.73	0.71	0.77	0.76	0.86	0.84
40	0.47	0.46	0.59	0.57	0.58	0.58	0.71	0.69
50	0.38	0.37	0.44	0.43	0.45	0.44	0.54	0.54
60	0.31	0.31	0.37	0.36	0.39	0.38	0.45	0.44
70	0.29	0.29	0.35	0.33	0.34	0.34	0.39	0.38
80	0.24	0.33	0.27	0.27	0.30	0.30	0.33	0.33
90	0.21	0.20	0.24	0.24	0.26	0.26	0.29	0.27
100	0.19	0.19	0.24	0.24	0.27	0.26	0.30	0.30

a composite estimator regardless of the sample size or the replacement time.

First Order Solutions

Another geometric approach to solving Blind Replacement uses the concave properties of $G(t)$ and associated first-order sufficient conditions expressed in equation (4.2). We can overlay, on a plot of the conditional mean life, $y = \int_0^t R(x)dx$, a plot of the function $y = (t+K)\{R(t)\}$. The latter function is the left hand side of equation (4.2) while the former is the right hand side of the equality. The solution to equation (4.2), then, is that t^* where these two plots cross as in Figure 11. Radner and Jorgenson (1962) show that equation (4.2) has a unique solution. Therefore, there can be only one crossing of these two plots. Since at $t = 0$, the left hand side of equation (4.2) is greater than the right hand side (as long as $K > 0$), it follows that the conditional mean life plot will cross the plot $y = (t+K)\{R(t)\}$ from below. Given a set of ordered lifetime data, $\{t_{(1)}, \ldots, t_{(n)}\}$, we can approximate both of these plots in Figure 11 with lines connecting each function evaluated at $t_{(i)}$, $i=1, \ldots, n$. First, we approximate the conditional mean life, $\int_0^{t_{(i)}} R(x)dx$, by the total time on test statistic divided by n. Then, the second function can be approximated as $y = (t_{(i)}+K)\{(n-i+1)/n\}$ using the California plotting position. Thus, we can estimate t^* by $t^@$ where $t^@$ is that

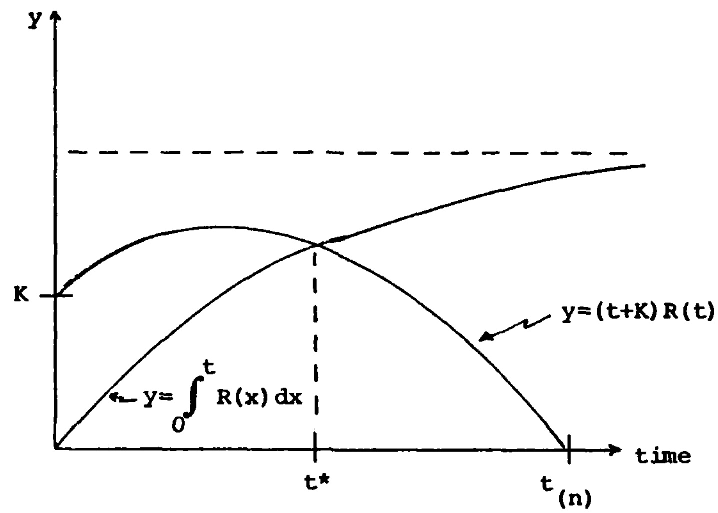

Fig. 11. Graphical First-Order Solution to Blind Replacement

$t_{(i)}$ closest to the crossing of the piece-wise linear approximations of both plots.

Since we approximate two functions with the first-order solution technique rather than one function as with a direct numeration solution, the first-order methodology is more unstable than direct numeration. This instability is apparent in a comparison of a first-order solution versus direct numeration as shown in Table 17. Accordingly, we do not recommend using a first-order solution technique.

Summary

The Blind Replacement model espoused by Radner and Jorgenson is a simple, probabilistic representation of the maintenance of a single-part item where the objective is to

TABLE 17

COMPARISON OF EXPECTED VALUES OF OPTIMAL BLIND REPLACEMENT ESTIMATORS FOR AN EXPONENTIAL DISTRIBUTION (MEAN = 200) USING A DIRECT NUMERATION (t*) TECHNIQUE VERSUS A FIRST ORDER (T@) SOLUTION TECHNIQUE

Sample Size	for Replacement Time, K = 10		400	
	E{T*}	E{T@}	E{T*}	E{T@}
10	67.2	79.6	309.7	361.1
20	63.9	70.3	306.8	330.4
30	62.7	67.2	304.6	319.8
40	61.9	65.2	303.3	315.4
50	61.7	64.4	303.0	311.8
60	61.3	63.5	303.1	310.9
70	61.1	63.0	302.2	308.7
80	61.1	62.8	301.8	307.6
90	60.8	62.4	301.5	306.4
100	60.6	62.0	302.2	307.0
∞	60.1	60.1	301.1	301.1

maximize the item's availability when the item's condition is unknown. This model is amenable to graphical solutions similar in nature to Bergman's technique for Age Replacement. We proposed, herein, several solution techniques and discussed the rapidity of convergence of each to the optimal Blind Replacement interval based on simulations from several common life distributions. We note that the proposed first-order solution is relatively unstable compared to direct numeration and, therefore, do not recommend its use. As for direct numeration solution techniques, we show that the use of the total time on test statistic leads to biased estimates of t^*. This bias might be reduced, for moderate replacement times, by using a combination of graphical estimators based on the total time on test statistic and the survivor time on test statistic.

CHAPTER V

CONFIDENCE BOUNDS IN MAINTENANCE PLANNING

Given a sample of failure data, the graphical techniques for Age, Block, and Blind Replacement, discussed in Chapters II, III, and IV, provide point estimates of the optimal replacement interval. Ideally, we would like to see confidence bounds about these point estimates in order to judge the degree of dispersion inherent in any sampling technique. However, for a single sample of size n, the distribution of the optimal replacement interval estimate, $t_{(j)} \approx t^*$, is that of the J^{th} order statistic where J is itself a random variable. Its sampling distribution is of an unknown form and is not amenable to straightforward statements of statistical variability. Thus, measures of dispersion, which gage sampling accuracy for small sample sizes, may have to be obtained by artificial sampling experiments. In this chapter, we discuss several techniques for bounding the true optimal replacement interval, t*, using a distribution free resampling concept known as the "percentile method." We illustrate this methodology for Age Replacement and Blind Replacement; however, this resampling concept is also appropriate for Block Replacement.

Efron (1982) discusses a resampling procedure known as the "bootstrap" and advocates the percentile method, a nonparametric confidence interval technique developed from the bootstrap procedure. The bootstrap is a computational process for estimating attributes of some general random variable, X, with an unspecified cumulative distribution function, F(x). This process involves repeated resampling of a single set of observations, $\{x_1, x_2, \ldots, x_n\}$, where n resample observations are uniformly drawn with replacement from the set of the original n observations. Each bootstrap resample set, $\{x_1^{\cdot}, x_2^{\cdot}, \ldots, x_n^{\cdot}\}$, can be used to compute some aspect of X, while repeated bootstrap resampling provides statistics on this aspect. In our case, we want to estimate t* from a set of n observed lifetimes, $\{t_{(1)} \leq \cdots \leq t_{(n)}\}$, and specify a confidence bound about t* without collecting more data. Accordingly, we can resample with replacement from the original lifetime data and accumulate b sets of bootstrap data, $\{t_{(1)}^{\cdot 1} \leq \cdots \leq t_{(n)}^{\cdot 1}\} \ldots \{t_{(1)}^{\cdot b} \leq \cdots \leq t_{(n)}^{\cdot b}\}$. For each of the b sets of bootstrap observations, we can compute $\hat{t}^{*i}$ (i=1, ..., b), the i^{th} bootstrap estimator of t*. The collection of b bootstrap estimators can themselves be ordered as $\{\hat{t}^{*(1)} \leq \cdots \leq t^{*(b)}\}$, and a 100 (1-α) percent confidence bound can be constructed from these ordered bootstrap estimators as $[\hat{t}^{*(\alpha b/2)}, \hat{t}^{*(b-\alpha b/2)}]$. We refer to this confidence bound as the "percentile method" bound.

In order to investigate the performance of the percentile method for Age Replacement and Blind Replacement, we turn to Monte Carlo simulation involving several distributions from the Weibull family whose true optimal replacement intervals, t*, are known a priori.

The Percentile Method for Age Replacement

We applied Efron's percentile method for confidence bounds to Bergman's graphical Age Replacement estimator, $\hat{t}^*$, based on 50 observations from selected IFR Weibull distributions, each with the same mean value. Our analysis is based on 1000 iterations of 50 original observations per iteration from the respective Weibull distributions. At each iteration, we resampled the 50 observations (an arbitrary) 100 times with a bootstrap sample size of 50. If the last ordered observation in the bootstrap sample minimized the objective function, then for numerical reasons we estimated t* as a large number in lieu of using infinity. We then constructed various confidence level bounds for each iteration, and noted whether the bound contained the true optimal Age Replacement interval, t*, an appropriate fraction of the time. The results from this simulation, summarized in Tables 18 through 20, indicate that, for some distributions and cost ratios, the percentile method may not provide confidence bounds that are wide enough to contain t* with the correct probability.

TABLE 18

PROBABILITY OF THE PERCENTILE METHOD CONFIDENCE INTERVAL CONTAINING THE AGE REPLACEMENT t* BASED ON 1000 ITERATIONS OF 50 OBSERVATIONS FROM SOME IFR WEIBULL DISTRIBUTIONS (WITH MEAN = 200)

Cost Ratio	Confidence Interval Probabilities for Weibull Distributions with Shape Parameter								
	1.5			2.0			3.0		
$C=C_2/(C_1-C_2)$	90%	80%	70%	90%	80%	70%	90%	80%	70%
.01	.312	.249	.198	.218	.160	.128	.169	.139	.116
1.0	.739	.645	.551	.936	.857	.769	.917	.830	.734
5.0	.938	.869	.805	.742	.611	.509	.764	.693	.597

TABLE 19

90% CONFIDENCE INTERVAL (CI) WIDTH BASED ON THE PERCENTILE METHOD FOR THE OPTIMAL AGE REPLACEMENT INTERVAL, t*, WITH 1000 ITERATIONS OF 50 OBSERVATIONS FROM SELECTED WEIBULL DISTRIBUTIONS (WITH MEAN = 200)

	Weibull Distribution with Shape Parameter					
	1.5		2.0		3.0	
$C=C_2/(C_1-C_2)$	t*	90% CI	t*	90% CI	t*	90% CI
.01	16	23-79	23	34-79	38	57-98
1.0	481	231-∞	250	175-∞	183	150-228
5.0	4341	406-∞	765	328-∞	335	254-∞

TABLE 20

AVERAGE MAXIMUM RELATIVE ERROR IN 90% CONFIDENCE INTERVAL WIDTH BASED ON THE PERCENTILE METHOD FOR THE OPTIMAL AGE REPLACEMENT INTERVAL, t*, WITH 1000 ITERATIONS OF 50 OBSERVATIONS FROM SELECTED WEIBULL DISTRIBUTIONS (WITH MEAN = 200)

Cost Ratio	Weibull Distribution with Shape Parameter					
	1.5		2.0		3.0	
$C=C_2/(C_1-C_2)$	C(t*)	% Error	C(t*)	% Error	C(t*)	% Error
.01	.1842	48.0	.0887	86.9	.0392	148.0
1.0	.0999	6.59	.0967	5.08	.0949	4.11
5.0	.0300	3.65	.0300	2.64	.0300	1.83

When there are large differences between the cost of failure and the cost of replacement (i.e a small cost ratio), the percentile method produces confidence bounds which have a low probability of containing t* and the potential for large relative error in the objective function. A partial explanation of this result involves the geometry of Bergman's technique which leads to the first-order statistic as an estimate of t* when the cost ratio is small. With scant original samples, it is possible that the smallest observed failure time is greater than t* thereby leading to an inability of the percentile method to capture t*. This difficulty can be diminished by increasing the original sample size. Thus, for small cost ratios, a prudent course of action would be to collect reasonably large numbers of observed failure times.

With moderate cost ratios (i.e. a moderate difference between the cost of failure and the cost of replacement), the percentile method performs adequately for Weibull distributions with shape parameters greater than or equal to 2.0. For large cost ratios (which implies a nominal difference between the cost of failure and the cost of replacement), the percentile method works well as long as the Weibull distribution is not too IFR. Although the 90 percent confidence intervals for these moderate to large cost ratios can be very wide (as shown in Table 19), the average maximum relative error in the objective function within these intervals is less than 10 percent.

The application of the percentile method to Age Replacement can result in a confidence bound that is nil. The likelihood of this possibility increases as the statistical confidence of the bound decreases. For the reasonably large confidence bounds (greater than 70 percent), we never encountered a zero width confidence bound throughout our simulation activity. Thus, for sufficiently large numbers of observations, say at least 50, there appears to be minimal risk of a zero range confidence bound as long as we consider confidence levels greater or equal to 70 percent.

The Percentile Method for Blind Replacement

We also applied Efron's percentile method for confidence bounds to our TTT statistic direct numeration estimator, $\hat{t}^*$, based on 50 observations from selected IFR Weibull distributions, each with the same mean value. Again our analysis is based on 1000 iterations of 50 original observations from the respective Weibull distributions. At each iteration, we resampled the 50 observations 100 times with a bootstrap sample size of 50, constructed various confidence bounds for each iteration, and noted whether the bound contained the true optimal Blind Replacement interval. The results from this simulation, contained in Table 21, indicate that the percentile method produces confidence bounds which are not quite wide enough to contain t^* with the correct probability.

TABLE 21

PROBABILITY OF THE PERCENTILE METHOD CONFIDENCE INTERVAL CONTAINING THE BLIND REPLACEMENT t* BASED ON 1000 ITERATIONS OF 50 OBSERVATIONS FROM SOME WEIBULL DISTRIBUTIONS (WITH MEAN = 200)

Replacement Time, K	Confidence Interval Probabilities for Weibull Distributions with Shape Parameter								
	1			2			3		
	90%	80%	70%	90%	80%	70%	90%	80%	70%
10	.880	.785	.677	.862	.756	.645	.881	.779	.676
50	.885	.776	.672	.881	.794	.691	.887	.776	.689
100	.859	.751	.660	.877	.786	.667	.883	.785	.676
200	.878	.778	.681	.867	.762	.676	.880	.771	.674
400	.874	.781	.688	.885	.788	.687	.872	.766	.671

The reason for this weak performance involves the positive bias in the Blind Replacement estimator $\hat{t}^*$ (using the total time on test statistic) which we noted in Chapter IV. One approach for improving this performance is to modify the percentile method by reducing the bootstrap resample size to some fraction of the original observed sample size. Decreasing the bootstrap resample size will increase the variability in our biased estimator and, ideally, will widen the confidence bound. Unfortunately, this approach does not work for Age Replacement as it does for Blind Replacement.

We replicated our Blind Replacement simulations with a modified percentile method involving bootstrap resample sizes of 80 percent of the original sample size in lieu of the 100 percent bootstrap resample size used above. In other words, each of the 100 bootstrap resampled data sets contained only 40 observations drawn with replacement from the original 50 observations. The results, depicted in Tables 22 through 24, are quite satisfactory. This modified percentile method produced confidence bounds which contained t^* approximately the correct number of times in 1000 iterations as can be seen in Table 22. The 90 percent confidence bounds given in Table 23 are sufficiently narrow to be able to bracket t^* to within a 10 percent maximum relative error in the objective function as shown in Table 24.

TABLE 22

PROBABILITY OF THE MODIFIED PERCENTILE METHOD CONFIDENCE INTERVAL CONTAINING THE BLIND REPLACEMENT t* BASED ON 1000 ITERATIONS OF 50 OBSERVATIONS FROM SOME WEIBULL DISTRIBUTIONS (WITH MEAN = 200)

Replacement Time, K	Confidence Interval Probabilities for Weibull Distributions with Shape Parameter								
	1			2			3		
	90%	80%	70%	90%	80%	70%	90%	80%	70%
10	.916	.829	.730	.892	.803	.693	.920	.836	.732
50	.919	.830	.728	.920	.849	.757	.929	.827	.736
100	.904	.807	.704	.912	.828	.734	.913	.826	.735
200	.912	.837	.738	.919	.815	.725	.915	.831	.725
400	.903	.831	.739	.910	.843	.741	.911	.828	.726

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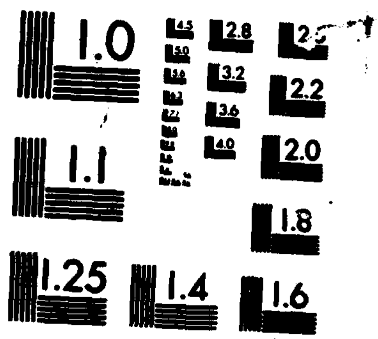

MICROCOPY RESOLUTION TEST CHART

TABLE 23

90% CONFIDENCE INTERVAL (CI) WIDTH BASED ON THE MODIFIED PERCENTILE METHOD FOR THE OPTIMAL BLIND REPLACEMENT INTERVAL, t*, with 1000 ITERATIONS OF 50 OBSERVATIONS FROM SELECTED WEIBULL DISTRIBUTIONS (WITH MEAN = 200)

Replacement Time, K	Weibull Distribution with Shape Parameter					
	1		2		3	
	t*	90% CI	t*	90% CI	t*	90% CI
10	60.1	45.1-83.6	89.3	71.7-114.5	41.9	29.1-67.0
50	126.5	97.9-163.5	147.2	123.8-173.8	113.9	78.5-172.2
100	176.6	134.3-218.0	180.0	153.5-207.5	172.7	117.4-264.1
200	229.3	177.9-289.3	217.2	187.1-247.7	258.9	171.9-406.9
400	301.1	231.2-385.2	257.5	221.0-292.7	383.1	245.8-618.6

TABLE 24

AVERAGE MAXIMUM RELATIVE ERROR IN 90% CONFIDENCE INTERVAL WIDTH BASED ON THE MODIFIED PERCENTILE METHOD FOR THE BLIND REPLACEMENT INTERVAL, t*, WITH 1000 ITERATIONS OF 50 OBSERVATIONS FROM SELECTED WEIBULL DISTRIBUTIONS (WITH MEAN = 200)

Replacement Time, K	Weibull Distribution with Shape Parameter					
	1		2		3	
	B(t*)	% Error	B(t*)	% Error	B(t*)	% Error
10	.7405	2.61	.8552	1.72	.5437	4.30
50	.5311	3.06	.6535	1.87	.3644	5.34
100	.4240	3.29	.5293	1.93	.2902	6.27
200	.3178	3.67	.3960	1.97	.1008	7.39
400	.2220	4.17	.2720	2.06	.0335	9.04

Summary

Nonparametric confidence bounds for the optimal replacement interval, t*, can be constructed via a resampling technique which Efron (1982) calls the percentile method. This method uses the order statistics generated from a bootstrap resampling scheme to form bounds with a given level of statistical confidence. A Monte Carlo simulation of graphical solution techniques for Age Replacement and Blind Replacement suggests that in many cases the percentile method may not produce bounds with the correct statistical confidence. Moreover, in some cases, the average maximum error in the objective function within the confidence bounds can be substantial. For Blind Replacement, however, a modified percentile method can produce bounds with satisfactory levels of statistical confidence and nominal relative errors in the objective function.

APPENDIX

LIST OF SPECIAL NOTATION

$E\{V\}$	Expected cost during a typical renewal cycle (p. 4)
$E\{L\}$	Expected renewal cycle length (p. 4)
$F(t)$	Failure distribution (p. 5)
$R(t)$	Survival distribution (p. 5)
$\mu(t)$	Conditional mean life (conditioned on censoring lifetimes at time t) (p. 5)
t^*	Optimal replacement interval (p. 6)
$\{t_{(1)}, \ldots, t_{(n)}\}$	Ordered observations where $t_{(i-1)} \leq t_{(i)}$ (p. 7)
$T(t_{(i)})$	Total time on test statistic (p. 7)
U_i	Scaled total time on test statistic (p. 8)
$S(t_{(i)})$	Survivor time on test statistic (p. 15)
N_i	Sample path of a renewal process (p. 16)
$M(t)$	Renewal function (p. 16)
$\hat{t}^*$	Estimate of t^* (p. 31)
$E\{T_{(i)}\}$	Expected value of the i^{th} order statistic (p. 32)
$\hat{E}\{T^*\}$	Expected value of the estimator of t^* (p. 39)
$m(t)$	Renewal density function (p. 43)
$\hat{t}^{*i}$	Estimate of t^* based on i^{th} bootstrap resample of n observed lifetimes (p. 75)

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